# Reputation: Speed vs. Accuracy

Sara Shahanaghi*

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**Abstract**

A sender dynamically learns about a state via conclusive signals and decides when to make a report. He seeks a reputation for having a high learning ability. In equilibrium, the sender faces an endogenous speed-accuracy tradeoff: accurate reports are better for reputation, but reputation suffers from delay. When the sender's learning ability is low, he increasingly prioritizes accuracy over speed as time passes, yet later reports can be less informative. Uninformed reports largely confirm the prior-likely state, and thus reports that are a priori unlikely are more informative.

*Toulouse School of Economics, 1 Esplanade de l'Université 31000 Toulouse, France. Email contact: sara.shahanaghi@tse-fr.eu. I would like to thank Navin Kartik for his guidance and support, and Nemanja Antic, Marco Ottaviani, and Maher Said for helpful feedback. Sara Shahanaghi acknowledges funding from ANR under grant ANR-17-EURE-0010 (Investissements d'Avenir program).

# 1. Introduction

Learning is often delegated to experts with career concerns, which can lead them to seek a reputation for being capable learners. Such experts can thus be prone to distorting the information they provide in an attempt to maximize their reputation. Indeed, Ottaviani and Sørensen (2006b) show that truthful information revelation is generically impossible under such a motive. However, experts often choose not only what to report, but *when* to do so. They may thus be concerned with how the timing of communication reflects their competence. An employee given a research task may consider how the length of time before reporting to his manager signals his aptitude for research. This is also true of news media and professional forecasters, who decide how quickly to publish news stories and call or predict election winners. In such settings, speed serves as a signal of competence, which in itself may lead to distortions in communication.

In this paper, we ask two questions. First, how does a reputation-driven sender distort communication if choosing not only what to report, but also *when* to do so? We are especially interested in understanding the dynamics of communication. Second, how is reputation formed in equilibrium? We consider how the passage of time affects the sender's reputation.

To this end, we present a model of a reputation-driven sender who dynamically learns and reports about a binary state. He learns via conclusive signals that arrive randomly and must decide when to report to the receiver. The sender may be either *good* or *bad*, where good senders have a higher ability to learn. His objective is to maximize his reputation, which is the receiver's posterior belief that he is good. This belief is formed by observing the sender's reporting behavior and some private signal about the state. We restrict attention to equilibria where the good sender reports truthfully in order to focus on the strategic reporting of the bad sender.

We first show that in any equilibrium, the bad sender persistently distorts his reporting: he fakes with positive probability in every period, where faking entails reporting despite being uninformed about the state. Specifically, when uninformed, the bad sender mixes between faking and truth telling, that is, abstaining from making a report. Although faking is persistent, the sender's reporting strategy is dynamic in equilibrium, as the probability of faking changes over time.

We then characterize equilibrium reputation to understand what drives this reporting behavior. We find that the sender is reputationally rewarded for two characteristics of his report: speed and accuracy. More precisely, reputation increases following accurate reports (and decreases following inaccurate ones), but deteriorates from delay. For an uninformed sender, these two objectives are at odds, giving rise to an endogenous speed-accuracy tradeoff: while faking prevents the reputational deterioration incurred from delay, abstaining allows the sender to learn in future periods and thus avert the reputational harm from inaccuracy. In equilibrium, the benefits of speed and accuracy counterbalance each other to preserve the uninformed sender's indifference between faking and truth telling. This demonstrates that even if a sender does not directly value speed or accuracy, in equilibrium he is induced to signal his ability precisely through these two channels.

Notably, the reputational cost from delay is not an artifact of the good sender's higher learning ability: it requires that the bad sender does not fake excessively in equilibrium, so that he is always less likely than the good sender to make a report. This result is driven by the equilibrium connection between reputation and faking. Specifically, at any given time, a higher probability of faking corresponds to a lower reputation from reporting. If delay were not harmful to reputation in some period, this would require sufficiently high faking in that period and thus, reporting in that period would yield a relatively low reputation. Since delaying would involve no immediate reputational decline, an uninformed sender would then prefer to wait rather than fake, giving rise to a profitable deviation.

This reputational cost of delay has important implications for the dynamics of the sender's reporting strategy. We show that if the bad sender's learning ability is sufficiently limited, faking dynamically falls. That is, as time passes, he relies increasingly on accuracy rather than speed to signal his ability. The reason lies in the role that speed and accuracy play in reputation formation. Because reputation suffers from delay, the sender must be compensated through some other means for abstaining in order to preserve equilibrium. There are two possible channels through which this compensation can occur. First, by abstaining, he may learn the state in future periods and thereby avoid the reputational harm from making an inaccurate report. The second channel is through the reputation function itself: if later reports incur a smaller penalty for inaccuracy, this incentivizes abstention.

Crucially, a smaller penalty for inaccuracy can only be consistent with a bad sender who fakes *less*. When the bad sender has limited learning ability, the second channel must prevail, implying that the probability of faking declines over time.

This decline in faking does not necessarily imply that later reports are more informative about the state. In fact, we show that later reports can be *less* informative. Declining informativeness occurs whenever the two types have sufficiently different learning abilities and is driven by a selection effect: delays in reporting are more likely for bad senders. This selection effect is strong enough that although the bad sender may be more truthful in later periods, later reports are more likely to be fake and thus less informative. Taken together, these results show that the dynamics of communication depend both on how the sender's reporting behavior evolves over time and on how delay changes the likelihood that the sender has high ability: it is for this reason that decreasing faking and decreasing informativeness can co-exist.

Finally, we consider how informativeness depends not only on timing, but also on the content of a report. In equilibrium, the bad sender is significantly more likely to fake the prior-likely state, and thus reports of the prior-unlikely state are more informative. This is despite the fact that accurate reports of the prior-unlikely state carry a larger reputational benefit. The high propensity for faking the likely state nonetheless occurs because such reports are more likely to be accurate from the perspective of an uninformed sender. This result implies that surprising reports are more credible in equilibrium.

**Related Literature** This paper contributes to two literatures: (1) reputation for learning and (2) signaling via action timing. In the literature on reputation for learning, an agent seeks to maximize an outsider's beliefs about his ability to learn. Ottaviani and Sørensen (2006b) present a general model of cheap talk under such a motive, showing that truthful communication is generically impossible. Other papers characterize the severity of such distortions. Ottaviani and Sørensen (2006c) show that a sender who observes a continuum of signals will send at most two messages in equilibrium, i.e., communication is highly coarse. Vong (2026) establishes conditions under which communication can improve the receiver's welfare. This paper contributes to the subset of the literature that qualitatively characterizes the nature of distortions (Scharfstein and Stein (1990), Ottaviani and

Sørensen (2006a), Prendergast and Stole (1996), Dasgupta and Prat (2008), Gentzkow and Shapiro (2006)). As in this paper, Prendergast and Stole (1996) and Dasgupta and Prat (2008) incorporate dynamic learning. In Prendergast and Stole (1996), a reputation-driven manager adjusts his level of investment while dynamically learning what the optimal level is. In contrast, we consider how the sender uses the *timing* of communication, rather than changes in his action, to affect beliefs about learning ability.

There is a vast literature studying signaling via decision timing. Bobtcheff and Levy (2017), Thomas (2019), and Halac and Kremer (2020) study firms that choose when to invest in or abandon a project and distort these decisions to signal high project quality. In Bobtcheff and Levy (2017) speed of investment can serve as either a positive or negative signal of quality depending on the firm's learning process. In Boleslavsky and Taylor (2024), an agent chooses when to submit a project to a principal where, as in this paper, distortions entail randomized faking that decreases over time. However, the faking agent faces a different tradeoff in their setting: he weighs the exogenous cost of delay against the endogenous probability that the project is rejected. A subset of this literature specifically studies the strategic timing of communication, as we do in this paper (Guttman, Kremer, and Skrzypacz (2014), Grenadier, Malenko, and Malenko (2016), Gratton, Holden, and Kolotilin (2018), Frug (2018), Deb, Pai, and Said (2025), Shahanaghi (2025)). Grenadier et al. (2016) and Frug (2018) study the timing of cheap talk communication by a sender who has biased preferences over the timing and direction of the receiver's decision, respectively. In Gratton et al. (2018), the sender is interested in affecting the receiver's beliefs about his type. They find that false reports are concentrated closer to the deadline, which stands in contrast to our finding that faking decreases over time. More substantively, this paper departs from this literature by considering a sender who wishes to affect beliefs about his learning ability. More similar to our environment, Deb et al. (2025) characterize the optimal contract for retaining an agent who learns and reports dynamically and also find that the agent is rewarded for both speed and accuracy. The distinction is that in their setting, these rewards are designed by the principal, whereas in ours they arise purely as an equilibrium means of signaling ability.

To our knowledge, there are three other papers that study the signaling of learning ability via timing. In Pant and Trombetta (2025), rival news firms are rewarded for reporting before their opponents but also have a reputational motive. They consider how reputation can discipline firms when competition incentivizes speed. In contrast, we consider a setting devoid of competition and show that reputation itself incentivizes speed, thereby deteriorating information quality. As in this paper, Smirnov and Starkov (2025) study a single reputation-driven sender who chooses when to report. They find that faking increases over time, but also that later reports are less informative. The most substantive difference concerns the reputational value of speed: while we find that speed is rewarded in equilibrium and document a speed-accuracy tradeoff, they find that *delay* improves reputation. This difference is driven by two modeling assumptions. First, while we assume the sender maximizes his expected reputation, they assume he holds highly convex, hence risk-loving, preferences over reputation. Second, we assume that the bad sender can learn, and this option value from learning drives the tradeoff between accuracy and speed. Finally, Shahanaghi (2026) studies a Wald problem with reputational motives, showing that speed can be rewarded or penalized in equilibrium and thus action can be premature or delayed.

The remainder of this paper proceeds as follows. In Section 2, we present the model. In Section 3, we characterize the equilibrium in a static version of the model. Section 4 characterizes the equilibrium of the full, dynamic model by building on the static characterization. Section 5 analyzes the equilibrium reputation function, establishing the speed-accuracy tradeoff. In Section 6, we characterize dynamics in reporting, while Section 7 studies how informativeness depends on the content of the report. Section 8 concludes. All formal proofs are relegated to the Appendix.

## 2. Model

There is one sender and one receiver. A binary, time-invariant state $\theta \in \{0,1\}$ is initially unknown to both players. At the beginning of the game, both sender and receiver hold prior $Pr(\theta = 1) = p_0$, where $p_0 \in [\frac{1}{2}, 1)$.

The sender has access to a private signal about the state, the informativeness of which depends on his type. The sender's type, denoted by $i$, may either be *good* ($i = G$) or *bad* ($i = B$). This type is private information: it is known to the sender, but not to the receiver. We let $R_0 \equiv Pr(i = G) \in (0, 1)$ denote the receiver's prior belief that the sender is good. The sender learns via conclusive Poisson signals: in any period $t \in \{1, 2, ..., T\}$, $\theta$ is privately revealed to the sender with probability $\lambda_i$. I.e., in each period, the sender observes a signal $r_t$:

$$r_t = \begin{cases} \theta & \text{with probability } \lambda_i \\ \emptyset & \text{with probability } 1 - \lambda_i. \end{cases}$$

where $r_t = \emptyset$ indicates that the state was not revealed to the sender at $t$. The $r_t$ are independent over time given $\theta$. We assume $\lambda_G > \lambda_B > 0$, namely that both types can learn, but the good sender learns the state more quickly than the bad sender.

At every period $t$ in which the game has not yet ended, after observing $r_t$, the sender may choose to make a *report*. A report consists of a message $m \in \{0, 1\}$ which is sent to the receiver. The sender can report at most once over the course of the game. Alternatively, he can choose to *abstain*, which consists of sending message $m = \emptyset$. The sender is not obligated to report, i.e., he can choose to abstain in every period. Let $m(t)$ denote the message sent at time $t$ by the sender. Let $\tau$ denote the time at which a report is made and denote the absence of a report by $\tau = \emptyset$. At $T + 1$ (i.e., after observing the sender's report, or lack thereof) the receiver observes a private signal $s \in \{0, 1\}$ about $\theta$ which is a Bernoulli draw. Specifically, $Pr(s = \theta) = \pi \in (\frac{1}{2}, 1)$. [1] The sender's payoff is given by his reputation, i.e., the receiver's belief that he is the good type, at the end of $T + 1$, given the time $\tau$ of report, the report $m(\tau)$ made by the sender (with $m(\emptyset) = \emptyset$) and the receiver's private signal $s$.

## 2.1. Equilibrium

A *Markov strategy* for a sender of type $i$ is a function $\sigma_t^i : \{0, 1, \emptyset\} \times [0, 1] \to [0, 1]$ for each $t \in \{1, ..., T\}$. Namely, $\sigma_t^i(m, p)$ denotes the probability that $i$ sends message $m$ at time $t$ under belief $p = \mathbb{P}[\theta = 1]$, conditional on not having reported before $t$. Since $\sigma_t^i$ specifies a distribution, we impose:

$$\sum_{m \in \{0,1,\emptyset\}} \sigma_t^i(m, p) = 1.$$

[1] We assume that this private signal is not perfectly correlated with the state to avoid the possibility of off-path occurrences where the sender is truthful but his report does not match the state.

By considering Markov strategies, we are restricting the strategy of a sender who has learned the state to not depend on the time $t$ in which he did so. For the equilibria we consider, this assumption is without loss and is used for notational convenience.

The receiver's beliefs about the sender are denoted by the *reputation function* $R \equiv (R_t(\cdot))_{t=1}^T$. $R_t(m,s)$ denotes the receiver's belief that the sender is good upon observing message $m \in \{0,1\}$ at time $t$ and private signal $s$. We let $R(\emptyset, s)$ denote the sender's reputation in the case that he never reports.

We seek a Markov perfect equilibrium. This consists of strategies $\sigma^B$ and $\sigma^G$ for the two types of sender as well as beliefs $p$ and $R$, where the strategies are optimal at every $t$ given the sender's private information and $R$, and beliefs ($p$ and $R$) are consistent with Bayes' Rule given the strategies, and in the case of the sender, his private information $r_1, ..., r_t$. Under the sender's information structure, at any given time he holds one of three different beliefs: he either knows the state (holds belief $p = 0$ or $p = 1$) and otherwise holds the prior belief $p_0$.

## 2.2. Selection

This game admits a multitude of equilibria. We impose two selection criteria to both refine the set of equilibria and to simplify the analysis. First, we restrict attention to equilibria where the good sender is *truthful*: he sends message $\theta$ if and only if he knows the state. This criterion is formalized as Selection Criterion 1.

**Selection Criterion 1.** *The good sender plays the truthful strategy in all $t$:*

$$\sigma_t^G(1,1) = \sigma_t^G(0,0) = \sigma_t^G(\emptyset, p_0) = 1.$$

By imposing that the good type does not report strategically, we are restricting attention to equilibria in which the bad type seeks to mimic the learning of the good type, rather than any possible manipulation in the good type's reporting. This criterion also aids in the analysis by ensuring that no messages are off-path in equilibrium.

Second, we impose that the bad sender who learns the state (i.e., holds belief $0$ or $1$) reports it immediately. However, unlike the good type, we do not impose that the uninformed bad type (i.e., one who holds belief $p_0$) is truthful. This is formalized as Selection Criterion 2.

**Selection Criterion 2.** *In all $t$, the bad sender truthfully reports the state if he has learned it:*

$$\sigma_t^B(1,1) = \sigma_t^B(0,0) = 1.$$

Selection Criterion 2 rules out reporting behavior in which the bad sender refrains from reporting the state once he has learned it purely out of indifference, as Selection Criterion 1 ensures that truthfully reporting the state must be optimal for a sender who knows the state, regardless of his type. This assumption serves the purpose of simplifying the analysis: it allows one to avoid the many technical complications associated with such potential indifferences.

Notably, these two criteria do not place restrictions on the senders' strategies: rather, we restrict attention to equilibria where such behavior is optimal for the senders. Hereafter, we refer to a Perfect Bayesian Equilibrium that satisfies the above two selection criteria as an equilibrium, and we will frequently refer to the bad sender simply as the sender.

**Faking and truth telling** Given the above selection criteria, the only behavior that one must characterize is that of the bad sender who is *uninformed*, i.e., who has not observed the state and thus holds belief $p_0$. We thus adopt two terms to describe the behavior of such an uninformed sender. He is *truthful* if he abstains when uninformed (sends message $m = \emptyset$), and he *fakes* if he makes a report when uninformed (sends message $m \in \{0, 1\}$).

# 3. Static equilibrium

Before analyzing the full dynamic model, we characterize the equilibrium in the static environment where $T = 1$. This elucidates certain features of equilibrium which will extend to the dynamic setting. For the remainder of this section, we will drop the time index from all objects.

## 3.1. Link between strategy and reputation

The analysis relies on the relationship between the sender's strategy and the reputation function, $R$. In the static environment, this relationship follows from Bayes' Rule as follows:

$$R(m,s) = \frac{1}{1 + L(m,s)\frac{1-R_0}{R_0}},$$

where $L(m,s)$ is the likelihood ratio of outcome $(m,s)$ for bad senders compared to good senders under $\sigma$:

$$L(m,s) \equiv \frac{\mathbb{P}(m,s|i=B)}{\mathbb{P}(m,s|i=G)}.$$ [2]

Thus, the reputation a receiver ascribes to a particular outcome depends on how likely the outcome is for the good sender as compared to the bad sender. In particular, any outcome that is relatively less (more) likely for the bad sender compared to the good sender will cause his reputation to improve (deteriorate) compared to his initial reputation $R_0$.

## 3.2. Equilibrium strategy and reputation

We first qualitatively characterize the bad sender's reporting strategy. Formally, we show that he cannot be fully truthful in equilibrium. Rather, he will mix between faking both reports ($m = 0$ and $m = 1$) and being truthful ($m = \emptyset$). This is stated as Claim 1. A proof is omitted as it is a special case of the dynamic characterization that will follow (Theorem 1).

**Claim 1** (Static equilibrium: faking)**.** *In any equilibrium, for all* $m \in \{0, 1, \emptyset\}$,

$$\sigma^B(m, p_0) > 0.$$

To see why the bad sender must fake with positive probability, let us suppose that he were to adopt a fully truthful strategy like the good sender. Because the bad sender learns the state with strictly lower probability, a good sender would be strictly more

[2] Formally, $\mathbb{P}(m,s|i)$ is a function of the sender's strategy, $\sigma^i$:

$$\mathbb{P}(m,s|i=B) = \sum_{\theta\in\{0,1\}} Pr(\theta)[\sigma^B(m,\theta)\lambda_B + \sigma^B(m,p_0)(1-\lambda_B)][\pi + (1-2\pi)\mathbb{I}(\theta \neq s)].$$

likely to report, and this holds for any realization of the receiver's private signal $s$. Meanwhile, abstaining ($m = \emptyset$) would be strictly more probable for the bad sender, and thus reputation-deteriorating. So, he can profitably deviate by faking. Precisely, both reports must be faked with positive probability: if only one report is faked, the one that is never faked would again guarantee an improved reputation, leading to a profitable deviation. Claim 1 also establishes that the uninformed bad sender does not always fake: he is truthful with positive probability. If he were to always fake, because the good sender abstains with strictly positive probability, abstaining would guarantee a perfect reputation, and thus would serve as a profitable deviation.

Next, we characterize equilibrium reputation. Namely, we show that the reputation function must satisfy two conditions. First, there is always a reputational reward for making an accurate report, and a penalty for making an inaccurate one. Formally, an *accurate* report is one that matches the receiver's private signal ($m = s$) and an *inaccurate* report is one which does not match ($m \neq s$). Second, we show that abstaining ($m = \emptyset$) always harms reputation. These two properties are stated as Claim 2.

**Claim 2** (Static equilibrium: reputation)**.** *In any equilibrium,*

1. *accuracy is rewarded: for any* $m \neq m' \in \{0, 1\}$

$$R(m, m) > R(m, m'),$$

2. *abstaining harms reputation:* $R(\emptyset, 0) = R(\emptyset, 1) < R_0$.

The reputational reward for accuracy and penalty for inaccuracy are driven by the equilibrium connection between the sender's strategy and the reputation function. Because the good type's reports are truthful, they are maximally correlated with $s$. However, this is not the case for the bad type: by Claim 1, he will report $m \in \{0, 1\}$ with strictly positive probability even when the state is not $m$, and his reports are consequently less correlated with $s$. It follows that reports by good (bad) senders are more likely to be accurate (inaccurate). The reputation function responds to this by ascribing a higher (lower) reputation to accurate (inaccurate) reports.

Meanwhile, abstaining strictly harms reputation because otherwise, an uninformed sender could guarantee an improvement to his reputation, implying a profitable deviation. Thus, the bad sender's reputation weakly improves in expectation. However, this would then imply that the reputation function is not consistent with Bayes' Rule: because the two types engage in different reporting behavior, there must be some learning about the sender's type in equilibrium, implying that the bad sender's reputation must strictly decrease in expectation.

### 3.3. Equilibrium existence and uniqueness

In the previous subsection, we established a set of necessary conditions for equilibrium. We now claim that there exists a unique equilibrium. To this end, for any $\mathbf{b} \equiv (b_0, b_1) \in \mathbf{B}$, where

$$\mathbf{B} \equiv \{(b_0, b_1) \in [0,1] \times [0,1] | b_0 + b_1 \leq 1\},$$

define strategy $\sigma^{\mathbf{b}}$ as follows:

$$\sigma^{\mathbf{b}}(1,1) = \sigma^{\mathbf{b}}(0,0) = 1 \qquad \sigma^{\mathbf{b}}(1,p_0) = b_1 \qquad \sigma^{\mathbf{b}}(0,p_0) = b_0.$$

Under such a strategy, the sender truthfully reports arrivals. Under non-arrival, he reports $1$ with probability $b_1$, reports $0$ with probability $b_0$, and abstains with the remaining probability. It follows from Selection Criterion 2 that the bad sender's strategy must take this form. We now claim that there exists a unique value of $b$ that constitutes an equilibrium. This is stated as Proposition 1.

**Proposition 1.** *When $T = 1$, there exists a unique equilibrium. Under this equilibrium, the bad sender plays strategy $\sigma^{\mathbf{b}^*}$, for some $\mathbf{b}^* \in int(\mathbf{B})$.*

The argument is most easily illustrated for $p_0 = \frac{1}{2}$. In this case, the sender must fake the two states with equal probability ($b_0 = b_1$), as otherwise, the reputation function would reward the report faked less often more highly, leading to a profitable deviation. So, finding an equilibrium entails finding a single value $b \equiv b_0 = b_1 \in (0, \frac{1}{2})$, and we let $\sigma^b$ denote the associated strategy.

Let us now consider how the reputation function, and consequently the value function, changes with $b$. Specifically, let $R^b(m,s)$ denote the reputation function that is consistent with strategy $\sigma^b$, and $V^b(m,p)$ the sender's value from sending message $m$ under belief $p$:

$$V^b(m,p) = qR^b(m,1) + (1-q)R^b(m,0),$$

where $q \equiv p\pi + (1-p)(1-\pi)$ is the probability that $s = 1$, given belief $p$ about the state. By Claim 1, in order for $\sigma^b$ to be played in equilibrium, indifference must be satisfied:

$$V^b(1, \frac{1}{2}) = V^b(0, \frac{1}{2}) = V^b(\emptyset, \frac{1}{2}).$$

When $b = 0$, i.e., when the bad sender never fakes, his lower arrival rate ($\lambda_B$) compared to the good sender ($\lambda_G$) implies that reporting will always strictly improve reputation, whereas abstaining will always harm reputation. Thus, reporting yields a strictly higher value for the uninformed sender:

$$V^b(1, \frac{1}{2}) = V^b(0, \frac{1}{2}) > V^b(\emptyset, \frac{1}{2}).$$

At the other extreme, when $b = \frac{1}{2}$, the bad sender never abstains. Because the good sender abstains with strictly positive probability, abstaining guarantees a perfect reputation. So, abstaining yields a strictly higher value for the uninformed sender:

$$V^b(1, \frac{1}{2}) = V^b(0, \frac{1}{2}) < V^b(\emptyset, \frac{1}{2}).$$

Thus, under $b = 0$, the uninformed sender strictly prefers faking, and under $b = 1/2$, he strictly prefers truth telling. At the same time, as $b$ increases, the reputation function responds by assigning increasingly higher (lower) reputation to senders who abstain (report). This is reflected in the value functions: the value of reporting strictly decreases in $b$, whereas the value of abstaining strictly increases in $b$ (Figure 1). Thus, there exists a unique point $b^* \in (0, 1/2)$ such that indifference is achieved.

## 3.4. Discussion

Under the static equilibrium the bad sender faces a reputational penalty both from inaccurate reporting and from abstaining. Importantly, the sizes of these reputational penalties are endogenous and closely tied to the level of faking in equilibrium: higher faking increases the reputational penalty from error, but

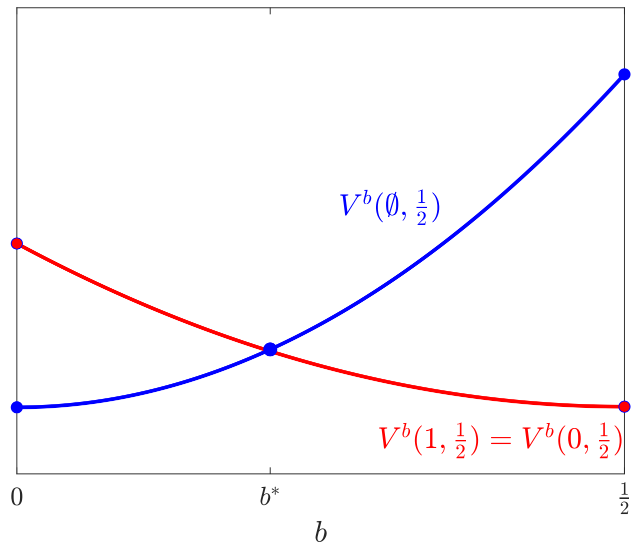


**Figure 1:** Value function associated with $\sigma^b$. The equilibrium faking probability $b^*$ is the unique $b$ such that indifference is satisfied.

decreases the penalty from abstaining. Indeed there is a unique level of faking that ensures these penalties are comparable and thus yields the uninformed sender indifferent between faking and truth telling, giving rise to an equilibrium. As we will discuss below, similar incentives prevail when $T \geq 2$. However, crucially, the *timing* of a sender's report serves as an additional signal of learning ability.

# 4. Dynamic equilibrium

We now characterize the structure of equilibrium strategies in the dynamic setting, showing that the bad sender mixes between faking and truth telling in every period. We will use this to establish the main results of the paper regarding dynamics in reputation and reporting in the sections that follow. In this section, we further establish equilibrium existence.

To this end, we begin by defining a $T$-period equivalent of the class of strategies considered in the static setting. Let $\mathbf{b} \equiv (\mathbf{b_1}, ..., \mathbf{b_T}) \in \mathbf{B}^T$, where $\mathbf{b}_t \equiv (b_t^0, b_t^1)$. For any such $\mathbf{b}$, let $\sigma^{\mathbf{b}}$ denote the strategy such that

$$\sigma_t^{\mathbf{b}}(1,1) = \sigma_t^{\mathbf{b}}(0,0) = 1 \qquad \sigma_t^{\mathbf{b}}(1,p_0) = b_t^1 \qquad \sigma_t^{\mathbf{b}}(0,p_0) = b_t^0.$$

Under this strategy, at each $t$, the sender truthfully reports arrivals and under non-arrival fakes $m$ with probability $b_t^m$. We now claim that any equilibrium, $\mathbf{b} \in int(\mathbf{B}^T)$; that is, the uninformed bad sender mixes between all three messages in every period. We further establish equilibrium existence.

**Theorem 1.** *There exists an equilibrium. In any equilibrium, the sender's strategy is given by $\sigma^{\mathbf{b}}$, for some $\mathbf{b} \in int(\mathbf{B}^T)$.*

The fact that the uninformed bad sender must mix between faking both messages and truth telling in any period follows from similar reasoning as in the static environment. Meanwhile, equilibrium existence follows from the Kakutani fixed point theorem.[3] Namely, we use the Kakutani fixed point theorem to show that there exist mixing probabilities $\mathbf{b}$ such that the reputation function consistent with this $\mathbf{b}$ induces indifference between faking and truth telling for the uninformed bad sender in every period. We then verify that under such a $\mathbf{b}$, the informed sender finds it optimal to truthfully report the state, thus establishing that this strategy must be consistent with an equilibrium.

While this class of strategies is a $T$-period extension of the static equilibrium strategy, it may be dynamic in nature. That is, the probability with which the sender fakes can evolve. Indeed, we will study such reporting dynamics in Section 6.

# 5. Equilibrium reputation

We now study the equilibrium reputation function, which illustrates the reporting incentives the sender faces in equilibrium. Our main finding is that the sender is reputationally rewarded for both speed and accuracy, and thus faces a speed-accuracy tradeoff.

## 5.1. Decomposing reputation

To this end, we decompose reputation into two components: one capturing the impact of timing and another capturing the impact of the report itself on reputation. In order to facilitate this decomposition, we begin by defining the sender's *interim*

[3] While we also established equilibrium uniqueness in the static environment, the argument does not extend to the dynamic setting.

*reputation*, denoted by $R_t$ for $t \in \{0, 1, ..., T\}$. This is the receiver's belief about the sender's type conditional on the event that he has not reported at or before $t$. Formally,

$$R_t \equiv \mathbb{P}[i = G | \tau \in \{t+1, ..., T, \emptyset\}].$$

$R_0$ is the sender's prior reputation, which is a parameter of the model. For all $t \geq 1$, $R_t$ is an equilibrium object, and is computed recursively using Bayes' Rule, given the sender's strategy. In equilibrium, it is a function of the bad sender's strategy and is given by the following:

$$R_t = \frac{1}{1 + \frac{1-R_{t-1}}{R_{t-1}} \frac{(1-\lambda_B)(1-b_t^1-b_t^0)}{1-\lambda_G}}.$$

While the sender is solely concerned with his reputation at the end of the game, $R_t$ captures how the sender's reputation dynamically evolves as the game progresses. Specifically, it specifies the sender's reputation at periods preceding that in which he reports. For instance, if the sender were to report in period $t = 5$, $R_4$ denotes his reputation immediately before his report was made.

By linearly separating the interim reputation from the reputation function, it can be written in the following form:

$$R_t(m, s) = R_{t-1} + \alpha_t^m \mathbb{I}(m = s) + \beta_t^m \mathbb{I}(m \neq s), \tag{1}$$

where, like $R_t$, $\alpha_t^m$ and $\beta_t^m$ are equilibrium objects and functions of the sender's strategy. Let us interpret this decomposition. The first component, $R_{t-1}$, denotes the sender's reputation immediately prior to his time $t$ report. This captures the effect that *delay* had on his reputation. Meanwhile, the residual component, $\alpha_t^m \mathbb{I}(m = s) + \beta_t^m \mathbb{I}(m \neq s)$, denotes the change that occurs in the sender's reputation from the report itself. This component accounts for the impact that a report, and its associated accuracy, has on reputation. Namely, $\alpha_t^m$ denotes the change in reputation due to an accurate report, whereas $\beta_t^m$ denotes the change from an inaccurate report. Both $\alpha_t^m$ and $\beta_t^m$ are contingent on time: the impact of accurate (or inaccurate) reporting on reputation can be dynamic. They are also contingent on the sender's report $m$: as we will show in Section 7, the effect of accuracy will in general depend on the report that was made.

## 5.2. Accuracy

We begin by characterizing the immediate impact of a report, and its accuracy, on reputation. Specifically, we show that an accurate report causes an increase in reputation compared to immediately prior to the report, whereas an inaccurate report causes a deterioration. This is formalized as Proposition 2.

**Proposition 2** (Accuracy and reputation). *In any equilibrium for any $m$ and $t$, $\alpha_t^m > 0 > \beta_t^m$.*

In part, this is an extension of Claim 2 from the static characterization: it implies that in any period, the sender's reputation from making an accurate report, $R_t(1, 1)$, strictly exceeds his reputation from making an inaccurate report, $R_t(1, 0)$. As in the static model, this is driven by the fact that faking, which is only done by the bad sender with positive probability, is associated with more inaccurate reporting.

However, Proposition 2 additionally asserts that the sender's reputation following an accurate (inaccurate) report must strictly exceed (lie below) his interim reputation immediately prior to that report. If both $\alpha_t$ and $\beta_t$ were positive, all time-$t$ senders could guarantee an improvement in their reputations by reporting. It would thus follow that at the end of the game, the sender's reputation will improve with probability $1$, regardless of his ability. Thus, the reputation function must not be consistent with Bayes' Rule, a contradiction. If instead both $\alpha_t$ and $\beta_t$ were negative, no sender would choose to report at time $t$, as it would cause his reputation to decline with probability $1$, even if his report is accurate. Thus, there would be a profitable deviation.

## 5.3. Speed

Next, we consider how speed affects reputation. We show that delay strictly damages reputation, i.e., $R_t$ is strictly decreasing. This is stated as Proposition 3.

**Proposition 3** (Speed and reputation). *In any equilibrium, $R_t$ is strictly decreasing in $t$.*

The strictly decreasing nature of $R_t$ follows from a backwards induction argument. That $R_T < R_{T-1}$ follows from identical reasoning as in the static setting (Claim 2). Namely, $R_T$ is the sender's reputation at the end of the game conditional on

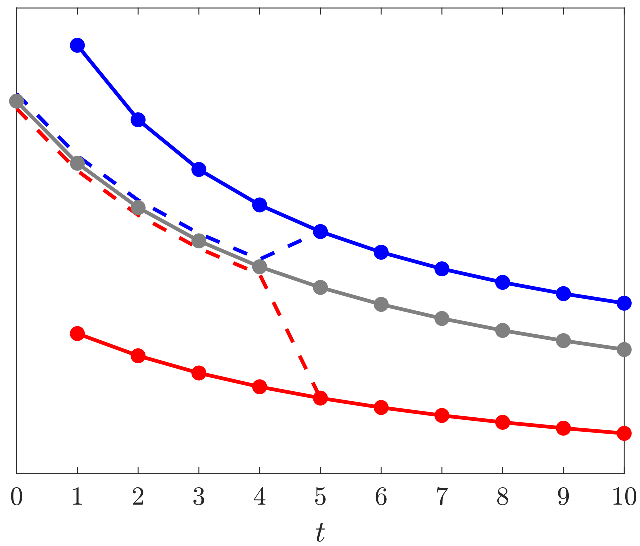


**Figure 2:** The grey line is the interim reputation ($R_t$). The solid blue (red) line is the reputation from making an accurate (inaccurate) report at $t$. The dotted red (blue) line shows dynamics of reputation for a sender who makes an inaccurate (accurate) report in period 5.

never having reported. So, if $R_T \geq R_{T-1}$, abstaining in $T$ is a "safe" message: an uninformed sender can guarantee a weak improvement to his reputation by abstaining. Thus, faking would never be optimal, giving rise to a profitable deviation.

Next, let us consider an arbitrary period $t$, and again assume by contradiction that $R_t \geq R_{t-1}$. Unlike in $T$, this does not imply that abstaining in $t$ is a safe message: since the game does not end in $t$, the sender's reputation may further decline after abstaining. In particular, even if the sender starts period $t+1$ with a relatively high reputation $R_t$, if $b^1_{t+1}$ or $b^0_{t+1}$ (i.e., the probability of faking either state) is sufficiently high, his value from reporting in the next period will be relatively low. This is because the more the bad sender reports on average (which is associated with a higher $b^1_{t+1}$ and $b^0_{t+1}$), the worse reporting in $t+1$ will be for his reputation, regardless of accuracy. Thus, to show that the sender can profitably deviate by abstaining in period $t$ when he is uninformed, we must show that the probability of faking in $t+1$ is sufficiently bounded.

In fact, such a bound follows from the inductive assumption that $R_t > R_{t+1}$. Formally, let $b_t$ denote the probability that the uninformed bad sender fakes either state in $t$:

$$b_t \equiv b_t^0 + b_t^1.$$

$R_t > R_{t+1}$ implies that $b_{t+1}$ lies below some threshold $\bar{b}$. [4]. This is the unique value of $b_{t+1}$ such that the good and bad sender abstain with equal probability, i.e., it is the solution to the following equality: $1 - \lambda_G = (1 - \lambda_B)(1 - b_{t+1})$. There is an intuition for this: the more the bad sender fakes (i.e., the higher $b_{t+1}$ is), the *better* abstaining must be for reputation, as it is less probable for the bad sender. This results in a higher $R_{t+1}$. Thus, $b_{t+1}$ must lie below this threshold to ensure that $R_{t+1}$ does not exceed $R_t$. This bound is sufficient to show that the uninformed sender at $t$ can profitably deviate by abstaining at $t$, thus showing that we cannot have $R_t \geq R_{t-1}$.

The dynamics of the sender's reputation, as well as the effects of both speed and accuracy, are shown in Figure 2, which illustrates our findings from this section. As shown, the sender's reputation is dynamically discounted until the time in which he reports (Proposition 3). Once he reports, his reputation will exhibit a strict upwards jump if his report was accurate, and a downwards jump if it was inaccurate (Proposition 2).

**Discussion** Together, these results imply that in equilibrium, the sender signals his learning ability through two means: speed and accuracy. For an uninformed sender, these two channels give rise to a tradeoff. Namely, by waiting to report later rather than reporting today, the uninformed sender may learn the state and thus improve his accuracy, but will suffer reputational harm from reduced speed. The reputational reward from speed is not merely an artifact of the good sender's higher learning ability. Rather, it is an equilibrium phenomenon: while the bad sender fakes in equilibrium, his level of faking in any period is sufficiently low that he is less likely to make a report than the good type, leading to a reputational deterioration from waiting.

## 6. Dynamics in reporting

In this section, we study dynamics in reporting. We first show that faking strictly decreases over time whenever the bad sender's learning ability is sufficiently limited.

[4] Formally, $\bar{b} \equiv \frac{\lambda_G - \lambda_B}{1 - \lambda_B}$

That is, the sender relies increasingly on accuracy rather than speed to signal his learning ability. We then show that despite this, later reports can be less informative.

## 6.1. Faking

We now state Proposition 4, which establishes that $b_t^m$ is strictly decreasing in $t$ provided that $\lambda_B$ lies below a threshold.

**Proposition 4** (Decreasing faking)**.** *Fixing all parameters except $\lambda_B$, there exists $\overline{\lambda}_B \in (0, \lambda_G)$ such that if $\lambda_B < \overline{\lambda}_B$, in any equilibrium $b_t^m$ is strictly decreasing in t for $m \in \{0, 1\}$.*

This result follows from the dynamics in the reputation function. Proposition 3 established that conditional on not having yet reported, the sender's reputation at the beginning of period $t+1$, $R_t$, must be strictly less than his reputation at the beginning of period $t$, $R_{t-1}$. So, all else equal, and in particular if the bad sender were to employ the same strategy in both periods (i.e., if $b_t^m = b_{t+1}^m$), reporting at time $t$ yields a strictly higher reputation than reporting at time $t+1$ regardless of whether the report was accurate. For an uninformed sender, this makes faking at time $t$ strictly advantageous compared to faking at $t+1$, as it yields a strictly higher reputation on average.

To ensure that the uninformed bad sender does not have a strict incentive to fake at time $t$, thus violating indifference, he must be compensated for waiting, which can happen through two separate channels. First, by waiting, he will with probability $\lambda_B$ learn the state in the next period, and increase his chances of making an accurate report. However, if $\lambda_B$ is sufficiently small, he must instead be compensated through a different channel: the reputation function. Although the sender starts off $t+1$ with a lower reputation, his marginal change in reputation from reporting in period $t+1$ ($R_{t+1}(m, s) - R_t$) must be strictly higher regardless of the realization of $s$. To see why this implies that $b_t^m > b_{t+1}^m$, we consider the relationship between $b_t^m$ and the reputation function. A higher $b_t^m$ is associated with a higher probability that the bad sender reports $m$, and thus the reputation function responds by assigning a *lower* reputation to reporting at time $t$ regardless of accuracy. Thus, a greater increase in reputation from reporting at $t+1$ can happen only if $b_t^m > b_{t+1}^m$.

This result shows that a bad sender of sufficiently low ability becomes more truthful over time, i.e., less inclined to fake a report. In other words, faking is most prevalent at the beginning of the learning process and gradually falls over time. As shown above, the sender has two separate means of signaling ability: speed and accuracy. It is because of these dual considerations that the uninformed sender finds it optimal to mix. While both considerations are present throughout the game, as time passes, he relies increasingly on accuracy rather than speed to signal ability, attempting to convince the receiver he is good through a growing reluctance for uninformed reporting.

## 6.2. Informativeness

We now show that despite the decreasing nature of faking, later reports can be less informative about the state. Precisely, this happens whenever the *good* sender has a sufficiently high learning ability.

To this end, we begin by defining informativeness. The *informativeness* of report $\theta$ at time $t$, $I_t^\theta$, is defined as follows:

$$I_t^\theta \equiv \mathbb{P}[r_t = \theta | m_t = \theta].$$

Recalling that $r_t = \theta$ denotes that the sender learns that the state is $\theta$ in time $t$, $I_t^\theta$ is the probability that a report $\theta$ at time $t$ is truthful. We call this informativeness because it denotes the probability with which a report at time $t$ reflects the sender's true signal, rather than the noise generated by faking.

In any equilibrium, informativeness can be computed using Bayes' Rule as follows:

$$I_t^\theta = \left(1 + \frac{(1-\lambda_B) b_t^\theta}{\mathbb{P}(\theta)(\lambda_G X_t + \lambda_B)}\right)^{-1}, \tag{2}$$

where $X_t \equiv \frac{R_{t-1}}{1-R_{t-1}}$ is the likelihood ratio of the interim reputation in the previous period and $\mathbb{P}(\theta)$ is the prior probability of $\theta$. This equation illustrates that the informativeness of a report is determined by two equilibrium objects: 1) the faking probability $b_t^\theta$ and 2) the interim reputation $X_t$. Specifically, informativeness is decreasing in the faking probability, and increasing in the probability that the agent is good conditional on not yet having reported. The latter is because only bad senders fake, and thus informativeness is increasing in the conditional probability that the agent is good. It then follows from (2) that for any $t < T$, $I_t^\theta > I_{t+1}^\theta$ if and only if

$$\frac{b^\theta_{t+1}}{b^\theta_t} > \frac{X_{t+1} + \lambda_B/\lambda_G}{X_t + \lambda_B/\lambda_G}.$$

Thus, even if faking is decreasing over time ($b^\theta_{t+1} < b^\theta_t$), later reports are less informative whenever this is outweighed by the rate of decline in the interim reputation. Indeed, by Proposition 3, such a decline in interim reputation always occurs in equilibrium. This illustrates an important point: changes in informativeness are not just determined by changes in faking, but also by a *selection effect*: in equilibrium, later reports are more likely to be made by bad senders.

We now show that if the good sender's learning ability is sufficiently high ($\lambda_G$ lies above a threshold), then this selection effect always outweighs any decrease in faking, making later reports less informative about the state. This is formalized as Proposition 5, where for analytical convenience, we restrict attention to a uniform prior ($p_0 = \frac{1}{2}$).

**Proposition 5** (Decreasing informativeness)**.** *Suppose $p_0 = \frac{1}{2}$ and fix all parameters except $\lambda_G$. There exists $\underline{\lambda}_G < 1$ such that if $\lambda_G > \underline{\lambda}_G$, then for all $\theta$ and all $t < T$:*

$$I^\theta_t > I^\theta_{t+1}.$$

A high $\lambda_G$ leads to decreasing informativeness because it ensures a sizeable selection effect while also requiring that the rate of faking not change substantially over time. Namely, when $\lambda_G$ is high, there is a large gap between the speeds at which the two types learn. As a result, delay must carry a large reputational cost in equilibrium, i.e., there must be a significant selection effect $R_{t-1} - R_t$. At the same time, this gap in learning abilities prevents the probability of faking, $b_t$, from falling below a threshold in any period. If it did, then a lack of report would strongly signal that the sender is bad, causing a significant reputational deterioration and leading to a profitable deviation. Thus, while $b^\theta_t$ can fall over time, it cannot fall too quickly, i.e., $\frac{b^\theta_{t+1}}{b^\theta_t}$ must remain relatively constant. The selection effect therefore dominates, leading informativeness to decrease over time.

**Discussion** Together, Proposition 4 and Proposition 5 imply that whenever the two types have sufficiently different learning abilities, in the sense that $\lambda_B$ is sufficiently low and $\lambda_G$ is sufficiently high, the bad sender becomes more truthful over time yet later reports are less informative about the state. This is driven by the fact that later reports select for bad senders, and this selection effect is so pronounced that it outweighs any decrease in faking, making later reports less informative. Notably, the fact that later reports select for bad senders is not merely an artifact of the gap between the senders' learning abilities, but rather due to the fact that delay must incur a substantial reputational cost to preserve equilibrium incentives.

# 7. Informativeness across reports

In this section, we consider how informativeness depends on the report being made. To illustrate the role of an asymmetric prior, we assume that $p_0 > \frac{1}{2}$, i.e., $\theta = 1$ is strictly more likely to realize a priori.

Before presenting our results regarding faking and informativeness, we make a useful observation about the reputation function: in any given period, there is a higher reputational reward from accurately reporting the ex-ante unlikely state ($\theta = 0$) than from accurately reporting the likely state ($\theta = 1$). Furthermore, the relative benefit of accurately reporting 0 increases as $\theta = 0$ becomes less likely. This is formalized as Corollary 1.

**Corollary 1.** *In any equilibrium, in any $t$:*

$$\frac{R_t(0,0) - R_t(1,0)}{R_t(1,1) - R_t(0,1)} = \frac{q_0}{1 - q_0},$$

*where $q_0 \equiv p_0\pi + (1 - p_0)(1 - \pi)$.*

This is an immediate corollary of the bad sender's indifference condition, and is driven by the fact that accuracy is rewarded in equilibrium, and from the perspective of an uninformed sender, a report of the a priori likely state (1) is more likely to be accurate. To maintain indifference between faking both messages, which is necessary in equilibrium, the reputational benefit from an accurate report under $s = 0$ must be higher than that under $s = 1$.

We now show that the state that is a priori more likely ($\theta = 1$) is faked with higher probability. More notably, conditional on faking, the sender reports the prior-likely state with probability greater than its prior probability. This implies that reports of the prior-improbable state are more informative. These findings are stated as Proposition 6.

**Proposition 6** (The prior and faking)**.** *In any equilibrium, for any $t$:*

$$\frac{b_t^1}{b_t^0} > \frac{p_0}{1-p_0},$$

*and thus $I_t^1 < I_t^0$.*

This result is closely connected to Corollary 1. For the reputation function and strategies to be consistent with each other in equilibrium, a lower benefit from reporting $1$ implies that the sender must fake $1$ more often than $0$. In fact, one can show that his relative probability of faking $1$ must be so much higher that a report is more likely to be $1$ conditional on being fake than it is conditional on being truthful: $\frac{b_t^1}{b_t^0} > \frac{p_0}{1-p_0}$. This in turn implies that $0$ reports are more likely to be truthful than $1$ reports, and hence more informative.

# 8. Conclusion

In this paper, we sought to understand how and why a motive to signal learning ability distorts information when the timing of communication is endogenous. By introducing a model in which senders learn dynamically via conclusive signals, we show that this reputational motive induces an endogenous speed-accuracy tradeoff wherein accuracy is reputationally beneficial but delay is harmful. This penalty of delay has implications for the dynamics of reporting: for a sender with limited learning ability, faking declines over time, meaning that the sender relies increasingly on accuracy rather than speed as time passes. Despite this, later reports can be less informative due to a selection effect, where the sender is increasingly likely to have low ability as time passes.

In our model, the sender is forward-looking with regard to reputation, a feature which drives the speed-accuracy tradeoff and dynamics in reporting. This stands in contrast to other dynamic environments (Prendergast and Stole (1996), Dasgupta and Prat (2008)) where agents are assumed to be myopic with regard to reputation. This paper illustrates how equilibrium outcomes can substantially differ when reputation is managed by a forward-looking sender rather than a myopic one. How a forward-looking agent behaves in more general environments, when not restricted to a timing decision, presents avenues for further research.

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# 9. Appendix

Before proceeding, we will introduce some notation and conventions. We begin with the sender's value function. Let $V_t^i(m, p)$ denote type-$i$ sender's value from sending message $m$ at time $t$ when he holds belief $p$. If $m = \emptyset$ this is equal to the sender's continuation value. If $m \in \{0,1\}$, this value is a linear function of the sender's belief $p$:

$$V_t^i(m,p) = qR_t(m,1) + (1-q)R_t(m,0)$$

for $m \in \{0,1\}$, where $q$ is the sender's belief that $s = 1$ will realize, formally $q \equiv p\pi + (1-p)(1-\pi)$. Let $V_t^i(p)$ denote the sender's value at time $t$ under belief $p$:

$$V_t^i(p) \equiv \max_m V_t^i(m,p).$$

In the analysis that follows, we will often drop the superscript when referring to the bad sender.

Next, let

$$L_t(m,s) \equiv \frac{\mathbb{P}(m_t = m, s|B, \tau \geq t)}{\mathbb{P}(m_t = m, s|G, \tau \geq t)},$$

where $\mathbb{P}(m_t = m, s|i, \tau \geq t)$ denotes the probability that $(\tau = t, m_t = m, s)$ will realize, given that a sender of type $i$ had not reported before time $t$. Note that in any equilibrium:

$$R_t(m,s) = \frac{1}{1 + \frac{1-R_{t-1}}{R_{t-1}} L_t(m,s)}.$$

Finally, let $q_0$ denote the prior probability that $s = 1$ realizes:

$$q_0 \equiv \pi p_0 + (1-\pi)(1-p_0).$$

**Proof of Claim 2.** We first show $R(1,1) > R(1,0)$. That $R(0,0) > R(0,1)$ follows symmetrically. By the expression for $R$, showing $R(1,1) > R(1,0)$ is equivalent to showing that $L(1,1) < L(1,0)$. It follows by Bayes' Rule that

$$L(1,1) = \frac{1}{\lambda_G p_0}[\frac{1-\pi}{\pi}\mathbb{P}(m = 1, \theta = 0|i = B) + \mathbb{P}(m = 1, \theta = 1|i = B)]$$

$$L(1,0) = \frac{1}{\lambda_G p_0}[\frac{\pi}{1-\pi}\mathbb{P}(m = 1, \theta = 0|i = B) + \mathbb{P}(m = 1, \theta = 1|i = B)],$$

It follows from Claim 1 that $\mathbb{P}(m = 1, \theta = 0|i = B) > 0$. Since by assumption $\pi > \frac{1}{2}$, it follows that $L(1,1) < L(1,0)$.

Meanwhile, 2 follows immediately from Proposition 3. $\square$

Before proceeding with the proof for Proposition 1, we introduce some notation. Recalling the definition of $\sigma^b$ above, let $R^b$, $V^b$, and $L^b$ denote the reputation, value and likelihood functions, respectively that are consistent with $\sigma^b$. Furthermore, let

$$X^\theta(b) \equiv V^b(\theta, p_0) - V^b(\emptyset, p_0),$$

and define $X(b) \equiv (X^0(b), X^1(b))$.

**Proof of Proposition 1.** We wish to show that there exists a unique $b$ such that $\sigma^b$ is an equilibrium strategy. Existence of such a $\sigma^b$ follows from the dynamic-model generalization (Theorem 1). It remains to show that this equilibrium is unique.

Recall that in any equilibrium, it follows from Theorem 1 that $X^0(b) = 0$ and $X^1(b) = 0$. We begin by showing that $X^0(b)$ and $X^1(b)$ are continuous and strictly decreasing in both $b^0$ and $b^1$. It suffices to show that $V^b(1, p_0)$ and $V^b(0, p_0)$ are continuous and weakly decreasing in both $b^0$ and $b^1$ and $V^b(\emptyset, p_0)$ is continuous and strictly increasing in $b^0$ and $b^1$. First note that for all $s \in \{0, 1\}$,

$$L^b(1, s) = \frac{(1 - \lambda_B)b^1\mathbb{P}(s) + \mathbb{P}(s|\theta = 1)\lambda_B p_0}{\mathbb{P}(s|\theta = 1)\lambda_G p_0}$$

which implies that $R^b(1, s)$ is continuous and weakly decreasing in $b^0$ and $b^1$ for $s \in \{0, 1\}$, and thus that $V^b(1, p_0)$ is continuous and weakly decreasing in $b^0$ and $b^1$. We can analogously show that $V^b(0, p_0)$ is continuous and weakly decreasing in both $b^0$ and $b^1$.

Next, note that

$$L^b(\emptyset, s) = \frac{(1 - \lambda_B)(1 - b^0 - b^1)}{(1 - \lambda_G)} \text{ for } s \in \{0, 1\}$$

which implies that $R^b(\emptyset, s)$, and consequently $V^b(\emptyset, p_0)$ is continuous and strictly increasing in $b^0$ and $b^1$ for $s \in \{0, 1\}$.

Now, suppose by contradiction that there exist $b$ and $\tilde{b}$ such that $b \neq \tilde{b}$ and both $b$ and $\tilde{b}$ constitute equilibria. By the above-established result that $X^0(b)$ and $X^1(b)$ are continuous and strictly decreasing in both $b^0$ and $b^1$, it follows that either

$$b^0 > \tilde{b}^0 \text{ and } b^1 < \tilde{b}^1$$

or

$$b^0 < \tilde{b}^0 \text{ and } b^1 > \tilde{b}^1.$$

Without loss of generality, suppose $b^0 > \tilde{b}^0$ and $b^1 < \tilde{b}^1$. Let $R$ ($\tilde{R}$) and $V$ ($\tilde{V}$) denote equilibrium reputation and value under equilibrium strategy $b$ ($\tilde{b}$), respectively. It follows that $R(0,s) < \tilde{R}(0,s)$ and $R(1,s) > \tilde{R}(1,s)$ for all $s$. Thus, since $V(1,p_0) = V(0,p_0)$,

$$q_0 R(1,1) + (1-q_0)R(1,0) = q_0 R(0,1) + (1-q_0)R(0,0).$$

Combined with the earlier inequalities, this implies

$$q_0 \tilde{R}(1,1) + (1-q_0)\tilde{R}(1,0) < q_0 \tilde{R}(0,1) + (1-q_0)\tilde{R}(0,0).$$

Thus, $\tilde{V}(1,p_0) < \tilde{V}(0,p_0)$. By Theorem 1, $\tilde{b}$ cannot constitute an equilibrium. Contradiction. □

We now state and prove a lemma.

**Lemma 1.** *In any equilibrium, $V_t(1,p)$ ($V_t(0,p)$) is weakly increasing (decreasing) in $p$. Furthermore, $V_t(1,p)$ ($V_t(0,p)$) is strictly increasing (decreasing) in $p$ whenever $\sigma_t(1,p_0) > 0$ ($\sigma_t(0,p_0) > 0$).*

**Proof.** We begin by proving that $V_t(1,p)$ is weakly increasing in $p$. All results for $V_t(0,p)$ follow symmetrically. First recall that

$$V_t(1,p) = p(2\pi - 1)[R_t(1,1) - R_t(1,0)] + [R_t(1,1)(1-\pi) + R_t(1,0)\pi]$$

Thus, to show that $V_t(1,p)$ is weakly increasing in $p$, it suffices to show that $R_t(1,1) \geq R_t(1,0)$. To show this, by definition of $R$, it suffices to show that $L_t(1,1) \leq L_t(1,0)$. This holds by definition, since

$$L_t(1,1) = \frac{p_0\lambda_B\sigma_t(1,1) + (1-p_0)\lambda_B\frac{1-\pi}{\pi}\sigma_t(1,0) + (1-\lambda_B)\sigma_t(1,p_0)(p_0 + (1-p_0)\frac{1-\pi}{\pi})}{p_0\lambda_G} \leq$$
$$\frac{p_0\lambda_B\sigma_t(1,1) + (1-p_0)\lambda_B\frac{\pi}{1-\pi}\sigma_t(1,0) + (1-\lambda_B)\sigma_t(1,p_0)(p_0 + (1-p_0)\frac{\pi}{1-\pi})}{p_0\lambda_G} = L_t(1,0)$$

Now, suppose $\sigma_t(1,p_0) > 0$. In this case, the above weak inequality on $L_t(1,1)$ and $L_t(1,0)$ will become a strict inequality, yielding that $R_t(1,1) > R_t(1,0)$, and thus that $V_t(1,p)$ is strictly increasing in $p$. $\square$

**Lemma 2** (Boundedness of $b_t$)**.** *Fix a $b \in int(\mathbf{B}^T)$. Let $R$ be consistent with $(\sigma^B = \sigma^b, \sigma^G)$, where $\sigma^G$ is the truthful strategy. Further, let $V$ denote the associated value function and suppose that for all $t$, $V_t^B(0,p_0) = V_t^B(1,p_0) = V_t^B(\emptyset,p_0)$. Then in every $t$:*

$$b_t < \frac{\lambda_G - \lambda_B}{1-\lambda_B},$$

*and thus $R_{t-1} > R_t$ for all $t \in \{1, ..., T\}$.*

**Proof.** Proof by induction. First, we show that

$$b_T < \frac{\lambda_G - \lambda_B}{1-\lambda_B}.$$

Suppose not, by contradiction. Then,

$$L_T(\emptyset, 1) = L_T(\emptyset, 0) = \frac{(1-\lambda_B)(1-b_T)}{1-\lambda_G} \leq 1,$$

where the inequality follows from the assumption on $b_T$. Thus,

$$R_T(\emptyset, 1) = R_T(\emptyset, 0) \geq R_{T-1},$$

and so $V_T(p_0) \geq V_T(\emptyset, p_0) \geq R_{T-1}$. However, since $V_T^B(1,p)$ is strictly increasing in $p$, $V_T^B(\emptyset, p)$ is constant in $p$, and by assumption $V_T^B(1,p_0) = V_T(\emptyset, p_0)$, it follows that

$$V_T^B(\emptyset, p_0) = V_T^B(\emptyset, 1) < V_T^B(1).$$

Likewise, one can show that $V_T^B(\emptyset, p_0) < V_T^B(0)$. It follows from the above inequalities that the bad sender's reputation at the end of $T$ must exceed his reputation at the beginning of $T$:

$$\lambda_B(1-p_0)V_T(0) + \lambda_B p_0 V_T^B(1) + (1-\lambda_B)V_T^B(p_0) > R_{T-1},$$

implying that $R$ is not consistent with Bayes' Rule given $(\sigma^B, \sigma^G)$. Contradiction.

Next, fix some $t < T$ and suppose by induction that $b_{t+1} < \frac{\lambda_G - \lambda_B}{1-\lambda_B}$. We wish to show

$$b_t < \frac{\lambda_G - \lambda_B}{1 - \lambda_B}.$$

Assume by contradiction that $b_t \geq \frac{\lambda_G - \lambda_B}{1-\lambda_B}$. This implies

$$R_t \geq R_{t-1}.$$

Further, for any $\rho$ and $m \in \{0, 1\}$,

$$b_\rho^m = b_\rho F_\rho(m),$$

where

$$F_\rho(m) = \mathbb{P}[m(\rho) = m | r_\rho = \emptyset, m(\rho) \neq \emptyset, \tau \geq \rho, i = B].$$

I.e., $F_\rho(m)$ is the probability that $B$ sends message $m$ conditional on faking at $\rho$. Since for any $\rho$, $F_\rho(1) + F_\rho(0) = 1$, for some $m \in \{0, 1\}$, $F_t(m) \geq F_{t+1}(m)$. Without loss of generality, suppose this $m = 1$. Now, by the definition of $R$:

$$R_t(1,1) = \frac{1}{1 + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{\lambda_B \pi p_0 + (1-\lambda_B) b_t F_t(1) q_0}{\lambda_G \pi p_0})}$$

$$R_{t+1}(1,1) = \frac{1}{1 + (\frac{1-R_t}{R_t})(\frac{\lambda_B \pi p_0 + (1-\lambda_B) b_{t+1} F_{t+1}(1) q_0}{\lambda_G \pi p_0})}.$$

It follows from the above inequalities on $b_t$ and $b_{t+1}$, combined with $R_t \geq R_{t-1}$, that

$$R_t(1,1) < R_{t+1}(1,1). \tag{3}$$

We can analogously show that

$$R_t(1,0) < R_{t+1}(1,0). \tag{4}$$

Next, note

$$V_t^B(\emptyset, p_0) \geq q_0 R_{t+1}(1,1) + (1 - q_0) R_{t+1}(1,0)$$

$$V_t^B(1, p_0) = q_0 R_t(1,1) + (1 - q_0) R_t(1,0).$$

Thus it follows from (3) and (4) that

$$V_t^B(\emptyset, p_0) > V_t^B(1, p_0).$$

Contradiction. Since

$$R_t = \frac{1}{1 + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{(1-\lambda_B)(1-b_t)}{1-\lambda_G})},$$

that $R_t < R_{t-1}$ follows immediately. □

**Proof of Theorem 1.** Now we establish existence of an equilibrium. We restrict attention to strategies of the form $\sigma^b$ for the bad sender, where $b \in \mathbf{B}^T$. For the good sender, we restrict attention to truthful strategies: for all $t$,

$$\sigma_t^G(1,1) = \sigma_t^G(0,0) = \sigma_t^G(\emptyset, p_0) = 1.$$

We now show that there exists a $b$ such that under the reputation function that is consistent with strategies $\sigma^b$ for $B$ and $\sigma^G$ for $G$, $\sigma_t^b(\emptyset, p_0)$ is optimal in all $t$. To this end, let $R^b$ denote the unique reputation function that is consistent with $\sigma^b$ and $\sigma^G$, given Bayes' Rule. Note that given $\sigma^G$ and that $\pi \in (\frac{1}{2}, 1)$, at every $t$ all $(m, s)$ pairs realize with positive probability, and thus $R^b$ is uniquely determined by Bayes' Rule for any $b$. Further let $V_t^{i,b}$ denote the value function for a sender of type $i$ given the reputation function $R^b$. We now proceed in a number of steps:

1. **Define the best-response correspondence:** Let us define $\Phi_t : \mathbf{B}^T \rightrightarrows \mathbf{B}$, where $\Phi_t(b)$ denotes the best response correspondence for an uninformed bad sender (a $B$ sender who holds belief $p_0$) at $t$ given reputation function $R^b$. Formally, $\Phi_t(b)$ denotes the set of $\tilde{b}_t \in \mathbf{B}$ that are optimal at time $t$ given reputation function $R^b$. Next, let

$$\Phi(\cdot) \equiv \Phi_1(\cdot) \times ... \times \Phi_T(\cdot).$$

2. **Show that $\Phi$ has a fixed point** We prove existence of a fixed point by invoking the Kakutani fixed point theorem. It is immediate that $\Phi_t(b)$ is convex and non-empty for all $b$. It follows that their Cartesian product $\Phi(b)$ is also convex and non-empty for all $b$. Further, it follows by the same reasoning as in the static environment that $V_t^{B,b}$ is continuous in $b$ for all $m \in \{1, 0, \emptyset\}$. Thus, $\Phi$ is

upper hemi-continuous in $b$. It follows from the Kakutani fixed point theorem that $\Phi$ has a fixed point.

3. **Show that any fixed point of $\Phi$ is interior** Formally, we claim that if $b$ is a fixed point of $\Phi$, then $b \in int(\mathbf{B}^T)$. Suppose not, by contradiction. Let $t$ denote the first period such that $b_t \notin int(\mathbf{B})$. First suppose $b_t^0 + b_t^1 = 1$. This implies that

$$V_t^{B,b}(\emptyset, p_0) = 1 > V_t^{B,b}(m, p_0)$$

for $m \in \{0, 1\}$. Thus $b_t$ is not optimal, and so $b$ cannot be a fixed point. Next, we want to show that for all $t, m$, $b_t^m > 0$. Proof by induction. Fix a $t$ and assume $b_{t'}^m > 0$ for all $t' > t$ and for all $m$. Now, suppose by contradiction that for some $m$, $b_t^m = 0$. Then, it must also be that $b_t^{m'} = 0$ where $m' \neq m$, because if not

$$V_t^{B,b}(m, p_0) > V_t^{B,b}(m', p_0),$$

and thus $b$ cannot be a fixed point. First consider the case where $t = T$. It follows from Bayes' Rule that

$$R_T^b(1, s) > R_T^b(\emptyset, s)$$

for all $s \in \{0, 1\}$. Thus, $V_T^b(1, p_0) > V_T^b(\emptyset, p_0)$, implying that $b$ cannot be a fixed point. Next consider the case where $t < T$. That $b_t^0 = b_t^1 = 0$, combined with the fact that $b_{t+1}^m > 0$ for all $m$ implies that $R_t^b < R_{t-1}^b$. This further implies that

$$R_t^b(m, 1) = R_t^b(m, 0) > R_{t+1}^b(m, s)$$

for any $m \in \{0, 1\}$, $s \in \{0, 1\}$. Thus again $V_t^b(1, p_0) > V_t^b(\emptyset, p_0)$. Contradiction.

4. **Show that for any fixed point $b$ of $\Phi$, $(\sigma^G, \sigma^b, R^b)$ is an equilibrium.** Fix any fixed point $b$ of $\Phi$. To show that $(\sigma^G, \sigma^b, R^b)$ is an equilibrium, it suffices to show that the strategies are optimal given $R^b$ (that $R^b$ is consistent with Bayes' Rule given the strategies holds by definition). Namely, one must verify that the following conditions hold:

   (a) Both $B$ and $G$ find it optimal to truthfully report arrivals:

$$V_t^{i,b}(1, 1) = V_t^{i,b}(1) \text{ and } V_t^{i,b}(0, 0) = V_t^{i,b}(0).$$

(b) The uninformed $B$ sender is indifferent between messages $1$, $0$, and $\emptyset$:

$$V_t^{B,b}(1, p_0) = V_t^{B,b}(0, p_0) = V_t^{B,b}(\emptyset, p_0).$$

(c) The uninformed $G$ sender finds it optimal to send message $m = \emptyset$:

$$V_t^{G,b}(\emptyset, p_0) \geq V_t^{G,b}(1, p_0) \text{ and } V_t^{G,b}(\emptyset, p_0) \geq V_t^{G,b}(0, p_0).$$

To establish (a), we prove $V_t^{i,b}(1, 1) = V_t^{i,b}(1)$ for all $t$ (that the statement holds for $m = 0$ follows analogously). Proof by induction. We first show that at $t = T$, the statement holds. It suffices to show that

$$V_T^{i,b}(1, 1) \geq V_T^{i,b}(\emptyset, 1) \text{ and } V_T^{i,b}(1, 1) \geq V_T^{i,b}(0, 1).$$

To show that these inequalities hold, first note that given $\sigma^G$ and $\sigma^b$, $R_T^b(\emptyset, 1) = R_T^b(\emptyset, 0)$, and thus $V_T^{i,b}(\emptyset, p)$ is constant in $p$. Meanwhile,

$$R_T^b(1, 1) > R_T^b(1, 0) \text{ and } R_T^b(0, 0) > R_T^b(0, 1),$$

and thus $V_T^{i,b}(1, p)$ is strictly increasing in $p$ while $V_T^{i,b}(0, p)$ is strictly decreasing in $p$. Because $b$ is interior (by 3),

$$V_T^{i,b}(1, p_0) = V_T^{i,b}(0, p_0) = V_T^{i,b}(\emptyset, p_0),$$

which implies by the monotonicities of the value functions above that

$$V_T^{i,b}(1, 1) > V_T^{i,b}(\emptyset, 1) > V_T^{i,b}(0, 1).$$

Thus, the statement holds. Next, consider some $t < T$. Suppose by induction that $V_{t+1}^{i,b}(1) = V_{t+1}^{i,b}(1, 1)$. We want to show that $V_t^{i,b}(1) = V_t^{i,b}(1, 1)$. Again it suffices to show that the following two inequalities hold:

$$V_t^{i,b}(1, 1) \geq V_t^{i,b}(\emptyset, 1) \tag{5}$$

$$V_t^{i,b}(1,1) \geq V_t^{i,b}(0,1). \tag{6}$$

As when $t = T$, to show (6), note that $V_t^{i,b}(1,p)$ is strictly increasing in $p$, while $V_t^{i,b}(0,p)$ is strictly decreasing in $p$. Since $b$ is an interior fixed point, $V_t^{i,b}(1,p_0) = V_t^{i,b}(0,p_0)$. It follows that $V_t^{i,b}(1,1) \geq V_t^{i,b}(0,1)$.

To show (5), suppose by contradiction that

$$V_t^{i,b}(1,1) < V_t^{i,b}(\emptyset,1).$$

Note by the inductive assumption, this implies that

$$V_t^{i,b}(1,1) < V_{t+1}^{i,b}(1,1).$$

Since $V_t^{i,b}(1,p_0) > V_{t+1}^{i,b}(1,p_0)$, this inequality can only hold if:

$$R_t^b(1,1) < R_{t+1}^b(1,1).$$

Recall by the definition of $R$, this holds if and only if:

$$(\frac{1-R_{t-1}^b}{R_{t-1}^b})(\frac{\lambda_B p_0 \pi + q_0 b_t^1(1-\lambda_B)}{\lambda_G p_0 \pi}) > (\frac{1-R_t^b}{R_t^b})(\frac{\lambda_B p_0 \pi + q_0 b_{t+1}^1(1-\lambda_B)}{\lambda_G p_0 \pi}). \tag{7}$$

By Lemma 2, $R_{t-1}^b > R_t^b$, and thus by (7), $b_t^1 > b_{t+1}^1$. This then implies that

$$\frac{\lambda_B p_0 \pi + q_0 b_{t+1}^1(1-\lambda_B)}{\lambda_B p_0 \pi + q_0 b_t^1(1-\lambda_B)} > \frac{\lambda_B p_0(1-\pi) + (1-q_0) b_{t+1}^1(1-\lambda_B)}{\lambda_B p_0(1-\pi) + (1-q_0) b_t^1(1-\lambda_B)}.$$

By (7),

$$(\frac{1-R_{t-1}^b}{R_{t-1}^b})(\frac{\lambda_B p_0(1-\pi) + (1-q_0) b_t^1(1-\lambda_B)}{\lambda_G p_0(1-\pi)}) > (\frac{1-R_t^b}{R_t^b})(\frac{\lambda_B p_0(1-\pi) + (1-q_0) b_{t+1}^1(1-\lambda_B)}{\lambda_G p_0(1-\pi)}), \tag{8}$$

and thus $R_t^b(1,0) < R_{t+1}^b(1,0)$. Combined with the fact that

$$R_t^b(1,1) < R_{t+1}^b(1,1),$$

it follows that

$$V_t^b(1, p_0) < V_{t+1}^b(1, p_0) \leq V_t^b(\emptyset, p_0),$$

which contradicts the fact that $b$ is an interior fixed point.

For (b), this follows immediately from the fact that $b$ is an interior fixed point of $\Phi$, implying that the uninformed $B$ player finds it optimal to send all three messages.

To establish (c), note that for all $t$:

$$V_t^{G,b}(1, p_0) = V_t^{B,b}(1, p_0) = V_t^{B,b}(\emptyset, p_0) \leq V_t^{G,b}(\emptyset, p_0),$$

where the second equality follows from (b), and the inequality follows from the fact that $\lambda_G > \lambda_B$, i.e., at any $t$ $G$ observes a Blackwell more informative signal in the continuation game, implying that at any $t$, $G$ has a weakly higher continuation value.

Step 4 above, combined with Step 1, implies that there exists an equilibrium. Furthermore, because under any equilibrium $b$ must be a fixed point of $\Phi$, and we have shown that any fixed point $b \in int(\mathbf{B}^T)$, the sender's strategy is given by $\sigma^b$ for some $b \in int(\mathbf{B}^T)$.

□

**Proof of Proposition 3.** Note by Theorem 1 that in any equilibrium, $b \in int(\mathbf{B}^T)$. Thus, for all $t$,

$$V_t^B(0, p_0) = V_t^B(1, p_0) = V_t^B(\emptyset, p_0).$$

The statement then follows immediately from Lemma 2. □

**Lemma 3.** *Fixing all parameters except $\lambda_B$, for any $\overline{\lambda}_B < \lambda_G$ and $t > 1$, there exists $\Delta_t > 0$ such that if $\lambda_B < \overline{\lambda}_B$, then in any equilibrium,*

$$R_{t-1} - R_t > \Delta_t.$$

**Proof.** Proof by (backwards) induction.

We begin by showing the statement holds for $t = T$. Suppose not, by contradiction. Then, for some parameters $(\lambda_G, R_0, p_0, \pi)$ and some $\overline{\lambda}_B < \lambda_G$, there exists a sequence $(\lambda_B^n)_{n=1}^{\infty}$ where $\lambda_B^n < \overline{\lambda}_B$ for all $n$, such that

$$\lim_{n\to\infty} R_{T-1}^n - R_T^n = 0, \tag{9}$$

where $R^n$ denotes some equilibrium reputation under $\lambda_B^n$. Further let superscript $n$ denote objects under this equilibrium.

First, note there exists $\underline{R} > 0$ such that $R_{T-1}^n > \underline{R}$ for all $n$. Specifically,

$$R_{T-1}^n > \frac{1}{1 + (\frac{1-R_0}{R_0})(\frac{1}{1-\lambda_G})^{T-1}} > 0. \tag{10}$$

Since for any $n$, the uninformed sender must be indifferent at $T$ between messages $\emptyset$ and $1$, and since $R_T^n(\emptyset, 1) = R_T^n(\emptyset, 0) = R_T^n$,

$$q_0 R_T^n(1,1) + (1-q_0) R_T^n(1,0) - R_T^n = 0.$$

It then follows from (9) that

$$\lim_{n\to\infty} [q_0 R_T^n(1,1) + (1-q_0) R_T^n(1,0) - R_{T-1}^n] = \lim_{n\to\infty} [V_T^n(1,p_0) - R_{T-1}^n] = 0. \tag{11}$$

Now, we show that (9) further implies that there exists $\underline{b} > 0$ such that $b_T^{1,n} > \underline{b}$ for all $n$. To show this, suppose not by contradiction. Then, there exists a subsequence $\{b_T^{1,k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} b_T^{1,k} = 0$. However, this implies $\lim_{k\to\infty} b_T^{0,k} = 0$ as well, because otherwise, for large enough $k$, the uninformed bad sender could profitably deviate by always reporting $1$ instead of $0$. However, since

$$R_T^k = \frac{1}{1 + (\frac{1-R_{T-1}^k}{R_{T-1}^k})(\frac{(1-\lambda_B^k)(1-b_T^{1,k}-b_T^{0,k})}{1-\lambda_G})},$$

this implies that (9) cannot hold. Contradiction.

Thus, since

$$R_T^n(1,1) = \frac{1}{1 + (\frac{1-R_{T-1}^n}{R_{T-1}^n})(\frac{\lambda_B^n p_0 \pi + (1-\lambda_B^n) b_T^{1,n} q_0}{\lambda_G p_0 \pi})}$$

$$R_T^n(1,0) = \frac{1}{1 + (\frac{1-R_{T-1}^n}{R_{T-1}^n})(\frac{\lambda_B^n p_0(1-\pi) + (1-\lambda_B^n) b_T^{1,n}(1-q_0)}{\lambda_G p_0(1-\pi)})},$$

it follows from (10) that there exists $K > 0$ such that for all $n$, $R^n_T(1,1) - R^n_T(1,0) > K$. Since $\pi > q_0$, there exists $N$ and $K' > 0$ such that for all $n > N$,

$$\pi R^n_T(1,1) + (1-\pi) R^n_T(1,0) - R^n_{T-1} = V^n_T(1,1) - R^n_{T-1} > K'. \tag{12}$$

Recall that for any $n$ the uninformed sender must also be indifferent between messages $\emptyset$ and $0$. Thus by analogous reasoning to the above (and instead using the inequality $\pi > (1-q_0)$), there exists $N$ and $K'$ such that for all $n > N$,

$$\pi R^n_T(0,0) + (1-\pi) R^n_T(0,1) - R^n_{T-1} = V^n_T(0,0) - R^n_{T-1} > K'. \tag{13}$$

Applying the Law of Iterated Expectations twice, we have:

$$\begin{aligned} R^n_{T-1} &= \mathbb{E}[\mathbb{P}_T[i=G] | \tau \geq T] \\ &= \mathbb{P}[r_T = 1 | \tau \geq T] V^n_T(1,1) + \mathbb{P}[r_T = 0 | \tau \geq T] V^n_T(0,0) + \mathbb{P}[r_T = \emptyset | \tau \geq T] V^n_T(1, p_0). \end{aligned}$$

Note there exists $\underline{p} > 0$ such that $\mathbb{P}[r_T = 1 | \tau \geq T] > \underline{p}$ for all $n$. However, given (11) and (12) and (13), this implies that there exists $N'$ such that for all $n > N'$,

$$R^n_{T-1} < \mathbb{P}[r_T = 1 | \tau \geq T] V^n_T(1,1) + \mathbb{P}[r_T = 0 | \tau \geq T] V^n_T(0,0) + \mathbb{P}[r_T = \emptyset | \tau \geq T] V^n_T(1, p_0).$$

Contradiction.

Next, fix a $t < T$ and suppose by induction that for all $s > t$, there exists $\Delta_s > 0$ such that whenever $\lambda_B < \overline{\lambda}_B$, $R_{s-1} - R_s > \Delta_s$ in any equilibrium. We want to show that there exists $\Delta_t$ such that $R_{t-1} - R_t > \Delta_t$ in any equilibrium whenever $\lambda_B < \overline{\lambda}_B$. Suppose not, by contradiction. Then there exists a sequence of equilibria, indexed by $n$, such that

$$\lim_{n \to \infty} R^n_{t-1} - R^n_t = 0. \tag{14}$$

Since $R^n_t - R^n_{t+1} > \Delta_{t+1}$ for all $n$, (14) implies that there exist an $N$, $m \in \{0,1\}$ and $\Delta_b > 0$ such that for all $n > N$,

$$b^{m,n}_t - b^{m,n}_{t+1} > \Delta_b. \tag{15}$$

Without loss of generality, suppose $m = 1$. Since

$$R_t^n(1,1) = \frac{1}{1 + (\frac{1-R_{t-1}^n}{R_{t-1}^n})(\frac{\lambda_B p_0 \pi + (1-\lambda_B) b_t^{1,n} q_0}{\lambda_G p_0 \pi})}$$

$$R_t^n(1,0) = \frac{1}{1 + (\frac{1-R_{t-1}^n}{R_{t-1}^n})(\frac{\lambda_B p_0 (1-\pi) + (1-\lambda_B) b_t^{1,n} (1-q_0)}{\lambda_G p_0 (1-\pi)})},$$

it follows from (14) and (15) that there exists $N'$ such that if $n > N'$,

$$R_t^n(1,1) < R_{t+1}^n(1,1) \text{ and } R_t^n(1,0) < R_{t+1}^n(1,0).$$

Thus, $\sigma_t^G(1,1) = 1$ cannot be optimal. Contradiction. $\square$

**Proof of Proposition 2.** First, we establish that for all $t$ and $m$, $\alpha_t^m > 0$. Suppose not, by contradiction. Without loss of generality, suppose $m = 1$. Let us begin by establishing a set of inequalities. First, note that

$$R_t(1,1) = R_{t-1} + \alpha_t^1 \leq R_{t-1}.$$

Separately,

$$R_t(1,1) > R_t(1,0).$$

This holds by identical reasoning to what is presented in Claim 2 (i.e., the static case). This implies that $R_t(1,0) < R_{t-1}$. Further, by Proposition 3,

$$R_t < R_{t-1}.$$

Next, we claim this implies that

$$R_t(0,1) < R_{t-1}.$$

Suppose not, by contradiction. This implies $R_t(0,1) \geq R_t(1,1)$. Since $R_t(0,0) > R_t(0,1)$, it follows that $V_t(0,1) > V_t(1,1)$. This implies $\sigma_t^G(1,1) = 0$ in equilibrium. Contradiction.

Now, let $\mathbb{E}_{t-1}[\mathbb{P}_t[i = G | s = 1, \tau \geq t]]$ denote the receiver's expected time-$t$ belief about the sender's type in time $t-1$ given $s = 1$ and that the sender has not yet reported. Note that

$$\begin{aligned}\mathbb{E}_{t-1}[\mathbb{P}_t[i=G|s=1,\tau\geq t]] &= \mathbb{P}_{t-1}[m_t=1|s=1,\tau\geq t]R_t(1,1)\\ +\mathbb{P}_{t-1}[m_t=0|s=1,\tau\geq t]R_t(0,1) &+ \mathbb{P}_{t-1}[m_t=\emptyset|s=1,\tau\geq t]R_t.\end{aligned} \tag{16}$$

Now, note that given the senders' strategies,

$$\mathbb{P}_{t-1}[m_t=\emptyset|s=1,\tau\geq t]>0.$$

It thus follows from (16) and the above inequalities that

$$\mathbb{E}_{t-1}[\mathbb{P}_t[i=G|s=1,\tau\geq t]]<R_{t-1}.$$

However, since

$$R_{t-1}=\mathbb{P}_{t-1}[i=G|s=1,\tau\geq t],$$

this is a violation of the Law of Iterated Expectations. Contradiction.

Now, we show $\beta_t^m<0$ for all $t,m$. Suppose not by contradiction. Again, without loss of generality suppose $m=1$. Thus, for some $t$, $\beta_t^1\geq 0$. Since $R_t(1,1)>R_t(1,0)$, this implies $\alpha_t^1>0$. Thus,

$$V_t^B(p_0)=V_t^B(1,p_0)=R_t(1,1)q_0+R_t(1,0)(1-q_0)=R_{t-1}+q_0\alpha_t^1+(1-q_0)\beta_t^1>R_{t-1},$$

where the first equality follows from Theorem 1. This implies that the uninformed bad sender's reputation strictly increases in $t$. Thus, $R$ must not be consistent with Bayes' Rule. Contradiction.

□

**Proof of Proposition 4.** Suppose by contradiction that the statement does not hold. Then for some parameters $(\lambda_G,\pi,p_0,R_0,T)$, $t<T$, $m\in\{0,1\}$, and some strictly decreasing sequence $(\lambda_B^n)_{n=1}^\infty$ where $\lim_{n\to\infty}\lambda_B^n=0$ and $\lambda_B^n<\lambda_G$, it holds that for all $n$:

$$b_t^{m,n}\leq b_{t+1}^{m,n},$$

where $(b_t^{m,n})_{t=1}^T$ denotes the bad sender's strategy under some equilibrium where $\lambda_B=\lambda_B^n$. Without loss of generality, suppose $m=1$. Let superscript $n$ denote equilibrium objects under this equilibrium. Now, note:

$$V_t^{B,n}(1,p_0) = q_0 R_t^n(1,1) + (1-q_0)R_t^n(1,0)$$

$$V_t^{B,n}(\emptyset,p_0) = \lambda_B^n p_0[\pi R_{t+1}^n(1,1) + (1-\pi)R_{t+1}^n(1,0)] + \lambda_B^n(1-p_0)[\pi R_{t+1}^n(0,0) + (1-\pi)R_{t+1}^n(0,1)] \\ +(1-\lambda_B^n)[q_0 R_{t+1}^n(1,1) + (1-q_0)R_{t+1}^n(1,0)].$$

It follows from Lemma 3 combined with the assumption that $b_t^{1,n} \leq b_{t+1}^{1,n}$ that there exists a $\Delta_t > 0$ such that if $n > N$,

$$R_t^n(1,1) - R_{t+1}^n(1,1) > \Delta_t \text{ and } R_t^n(1,0) - R_{t+1}^n(1,0) > \Delta_t.$$

Since $\lim_{n\to\infty} \lambda_B^n = 0$, it follows from the above expressions for $V_t^{B,n}(1,p_0)$ and $V_t^{B,n}(\emptyset,p_0)$ that for some $n$,

$$V_t^{B,n}(1,p_0) > V_t^{B,n}(\emptyset,p_0).$$

However, by Theorem 1, this cannot hold in equilibrium. Contradiction.

□

**Proof of Proposition 5.** We proceed by stating and proving two intermediate claims.

1. **Fixing all parameters except $\lambda_G$, for any $\underline{\lambda}_G > \lambda_B$, there exists $\Delta_t > 0$ such that if $\lambda_G > \underline{\lambda}_G$, then**
$$\frac{X_t + \lambda_B/\lambda_G}{X_{t+1} + \lambda_B/\lambda_G} > \Delta_t + 1.$$
Note that since in any equilibrium $R_t \leq R_0$ (by Proposition 3), $X_t < X_1 = \frac{R_0}{1-R_0}$. Thus to prove the statement, it suffices to show that there exists $\Delta_t > 0$ such that if $\lambda_G > \underline{\lambda}_G$,
$$R_{t-1} - R_t > \Delta_t.$$
To establish this, we follow the same argument as the proof of Lemma 3, with two modifications. As in the proof of Lemma 3, we proceed by induction. Fix a $\underline{\lambda}_G > \lambda_B$. First, we show that there exists $\Delta_T$ such that if $\lambda_G > \underline{\lambda}_G$, then in any equilibrium
$$R_{T-1} - R_T > \Delta_T.$$

Suppose not, by contradiction. Then there exists a sequence $(\lambda_G^n)_{n=1}^{\infty}$, where $\lambda_G^n > \underline{\lambda}_G$ such that

$$\lim_{n\to\infty} R_{T-1}^n - R_T^n = 0, \tag{17}$$

where $R_t^n$ denotes some equilibrium reputation under $\lambda_G^n$. We now extend two steps of the proof of Lemma 3 to the current setting. The remainder of the proof remains unchanged.

- First we extend the claim that there exists $\underline{R} > 0$ such that $R_{T-1}^n > \underline{R}$ for all $n$. Suppose not, by contradiction. Then, there exists an infinite subsequence $(R_{T-1}^k)_{k=1}^{\infty}$ of $(R_{T-1}^n)_{n=1}^{\infty}$ such that

  $$\lim_{k\to\infty} R_{T-1}^k = 0.$$

  It follows that for all $s, m$

  $$\lim_{k\to\infty} R_T^k(m, s) = 0.$$

  However, in order for the $R^k$ to be consistent with Bayes' Rule, for all $k$,

  $$\max_{t<T,s,m} R_t^k(m, s) > R_0.$$

  Thus there exists $K$ such that for all $k > K$, it cannot be optimal for $G$ to play the truthful strategy. Contradiction.

- Next, we extend the claim that there exists $\underline{b} > 0$ such that $b_T^{1,n} > \underline{b}$ for all $n$. This follows from (17), combined with the fact that $\lambda_G^n > \underline{\lambda}_G$ for all $n$.

Suppose by induction that for all $s > t$, there exists $\Delta_s > 0$ such that $R_{s-1} - R_s > \Delta_s$. We want to show that there exists $\Delta_t > 0$ such that $R_{t-1} - R_t > \Delta_t$. The proof for this follows identically as in Lemma 3.

2. **Show that, fixing all other parameters and a $t$, for any $\Delta_t > 0$, there exists $\underline{\lambda}_G < 1$ such that if $\lambda_G > \underline{\lambda}_G$, then for any $\theta \in \{0, 1\}$,**

   $$\frac{b_t^\theta}{b_{t+1}^\theta} < 1 + \Delta_t$$

   **in any equilibrium.**

Suppose not, by contradiction. Then, there exists $t$, $\Delta_t > 0$, a sequence $(\lambda_G^n)_{n=1}^\infty$ such that $\lim_{n\to\infty} \lambda_G^n = 1$ and a sequence of associated equilibria in which

$$\frac{b_t^{\theta,n}}{b_{t+1}^{\theta,n}} \geq 1 + \Delta_t \tag{18}$$

for all $n$ and some $\theta \in \{0,1\}$. Note that under the assumption that $p_0 = \frac{1}{2}$, for all $\theta$,

$$b_t^{\theta,n} = \frac{b_t^n}{2},$$

where $b_t^n \equiv b_t^{1,n} + b_t^{0,n}$. Thus (18) is equivalent to

$$\frac{b_t^n}{b_{t+1}^n} \geq 1 + \Delta_t$$

for all $n$. However, now we claim that for all $t$

$$\lim_{n\to\infty} b_t^n = 1.$$

Suppose not, by contradiction. Then for some $t$, $\varepsilon > 0$, and subsequence $(b_t^k)_{k=1}^\infty$ of $(b_t^n)_{n=1}^\infty$, $b_t^k < 1 - \varepsilon$ for all $k$. Since $\lim_{k\to\infty} \lambda_G^k = 1$,

$$\lim_{k\to\infty} R_t^k = \lim_{k\to\infty} \frac{1}{1 + (\frac{1-R_{t-1}^k}{R_{t-1}^k})(\frac{(1-\lambda_B)(1-b_t^k)}{1-\lambda_G^k})} = 0.$$

However, this contradicts the above-established fact that for all $t$ and $n$, there exists $\underline{R} > 0$ such that $R_t^n > \underline{R}$.

Since $\lim_{n\to\infty} b_t^n = 1$ for all $t$, it follows immediately that

$$\lim_{n\to\infty} \frac{b_t^n}{b_{t+1}^n} = 1.$$

Contradiction.

Now, fix all parameters except $\lambda_G$, and fix any $t < T$. Choose any $\underline{\lambda}_G > \lambda_B$. It follows from 1 above that there exists $\Delta_t$ such that

$$\frac{X_t + \lambda_B/\lambda_G}{X_{t+1} + \lambda_B/\lambda_G} > 1 + \Delta_t$$

in any equilibrium when $\lambda_G > \underline{\lambda}_G$. Further, by 2 above there exists $\underline{\lambda}'_G \in (\underline{\lambda}_G, 1)$ such that for all $\theta$, in any equilibrium where $\lambda_G > \underline{\lambda}'_G$,

$$\frac{b_t^\theta}{b_{t+1}^\theta} < 1 + \Delta_t.$$

Thus, there exists $\underline{\lambda}'_G \in (\lambda_B, 1)$ such that in any equilibrium where $\lambda_G > \underline{\lambda}'_G$,

$$\frac{b_t^\theta}{b_{t+1}^\theta} < \frac{X_t + \lambda_B/\lambda_G}{X_{t+1} + \lambda_B/\lambda_G},$$

and thus $I_t^\theta > I_{t+1}^\theta$. $\square$

**Proof of Proposition 6.** Fix any equilibrium. First, we want to show that for all $t$,

$$\frac{b_t^1}{b_t^0} > \frac{p_0}{1-p_0}. \tag{19}$$

Fix an equilibrium and a $t$. Suppose by contradiction that

$$b_t^1 = p_0 X \text{ and } b_t^0 \geq (1-p_0)X$$

for some $X > 0$. Now, recall

$$q_0 R_t(1,1) + (1-q_0)R_t(1,0) = V_t(1,p_0) = V_t(0,p_0) = q_0 R_t(0,1) + (1-q_0)R_t(0,0),$$

where the second equality follows from the bad sender's indifference condition. It follows from Bayes' rule that:

$$q_0 R_t(1,1) = [\frac{1}{q_0} + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{\lambda_B/q_0 + X(1-\lambda_B)/\pi}{\lambda_G})]^{-1}$$

$$(1-q_0)R_t(0,0) \leq [\frac{1}{1-q_0} + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{\lambda_B/(1-q_0) + X(1-\lambda_B)/\pi}{\lambda_G})]^{-1}$$

$$(1-q_0)R_t(1,0) = [\frac{1}{1-q_0} + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{\lambda_B/(1-q_0) + X(1-\lambda_B)/(1-\pi)}{\lambda_G})]^{-1}$$

$$q_0 R_t(0,1) \leq [\frac{1}{q_0} + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{\lambda_B/q_0 + X(1-\lambda_B)/(1-\pi)}{\lambda_G})]^{-1}.$$

Since $q_0 > (1-q_0)$ and $\pi > (1-\pi)$, $V_t(1,p_0) > V_t(0,p_0)$. Contradiction.

Next, we want to show that

$$I_t^1 < I_t^0. \tag{20}$$

By the Law of Iterated Expectations, for any $\theta \in \{0,1\}$:

$$\begin{aligned} I_t^\theta &= \mathbb{P}[i=G|m=\theta,\tau=t]\mathbb{P}[r_t=\theta|i=G,m=\theta,\tau=t] \\ &\quad + \mathbb{P}[i=B|m=\theta,\tau=t]\mathbb{P}[r_t=\theta|i=B,m=\theta,\tau=t]. \end{aligned}$$

Now, note

$$\mathbb{P}[r_t=1|i=G,m=1,\tau=t] = \mathbb{P}[r_t=0|i=G,m=0,\tau=t] = 1,$$

and

$$\begin{aligned} \mathbb{P}[r_t=1|i=B,m=1,\tau=t] &= \frac{\lambda_B p_0}{\lambda_B p_0 + (1-\lambda_B)b_t^1} \\ < \frac{\lambda_B(1-p_0)}{\lambda_B(1-p_0)+(1-\lambda_B)b_t^0} &= \mathbb{P}[r_t=0|i=B,m=0,\tau=t], \end{aligned} \tag{21}$$

where the equalities follow from Bayes' Rule and the inequality follows from (19). Further note

$$\begin{aligned} \mathbb{P}[i=G|m=1,\tau=t] &= \frac{\lambda_G p_0 R_{t-1}}{\lambda_G p_0 R_{t-1} + (\lambda_B p_0 + (1-\lambda_B)b_t^1)(1-R_{t-1})} \\ < \frac{\lambda_G(1-p_0)R_{t-1}}{\lambda_G(1-p_0)R_{t-1} + (\lambda_B(1-p_0)+(1-\lambda_B)b_t^0)(1-R_{t-1})} &= \mathbb{P}[i=G|m=0,\tau=t] \end{aligned} \tag{22}$$

where the inequality again follows from (19). (20) then follows from (21) and (22).

□