Online Appendix

Proof of Lemma 1. Proof by induction. When $t = T$,

$$V_T^G(p, \emptyset) = pR(\emptyset, 1) + (1-p)R(\emptyset, 0),$$

which is linear in p and thus convex in p. Next, fix any $t < T$, and assume that $V_{t+1}(p, \emptyset)$ is convex in p. We want to show that $V_t(p, \emptyset)$ is convex in p. This is equivalent to showing that for all $p, p' \in [0,1]$ and $\lambda \in [0,1]$,

$$\lambda V_t(p, \emptyset) + (1-\lambda)V_t(p', \emptyset) \geq V_t(\overline{p}),$$

where $\overline{p} \equiv \lambda p + (1-\lambda)p'$. To this end, fix a p, p', and $\lambda \in [0,1]$ and define the following binary signal $b \in \{0,1\}$ on θ:

$$Pr(b = 1|\theta = 1) = \frac{p\lambda}{\overline{p}}, \qquad Pr(b = 1|\theta = 0) = \frac{(1-p)\lambda}{1-\overline{p}},$$

Now define the following two signals σ and $\tilde{\sigma}$:

$$\sigma : \{0,1\} \to \Delta([0,\infty]), \text{ where } \sigma(\theta) = F(\cdot|\theta)$$

$$\tilde{\sigma} : \{0,1\} \to \Delta([0,\infty] \times \{0,1\}), \text{ where } \tilde{\sigma}(\theta) = \tilde{F}(\cdot,\cdot|\theta),$$

and for $b^* \in \{0,1\}$, $\tilde{F}(s, b^*) = F(s|\theta)Pr(b \leq b^*|\theta)$. Note that $\tilde{\sigma}$ Blackwell dominates σ. Now assuming that the agent has prior belief p let $G_t(\cdot|p)$ and $\tilde{G}_t(\cdot|p)$ denote the distribution of posteriors after observing σ and $\tilde{\sigma}$, respectively. It follows from the Law of Iterated Expectations that:

$$\tilde{G}_t(q|\overline{p}) = Pr(b = 1|\overline{p})G_t(q|p) + Pr(b = 0|\overline{p})G_t(q|p') = \lambda G_t(q|p) + (1-\lambda)G_t(q|p'). \quad (19)$$

Now, note that since $V_{t+1}(p, \emptyset)$ is convex in p and $V_{t+1}(p, 1) = pR(t+1, 1) + (1-p)R(t+1, 0)$ is linear in p, $V_{t+1}(p) = \max\{V_{t+1}(p, \emptyset), V_{t+1}(p, 1)\}$ is also convex in p.

Thus

$$\begin{aligned} V_t(\overline{p}, \emptyset) &= \int_0^1 V_{t+1}(q) dG_t(q|\overline{p}) \\ &\leq \int_0^1 V_{t+1}(q) d\tilde{G}_t(q|\overline{p}) \\ &= \lambda \int_0^1 V_{t+1}(q) dG_t(q|p) + (1-\lambda) \int_0^1 V_{t+1}(q) dG_t(q|p') \\ &= \lambda V_t(p, \emptyset) + (1-\lambda) V_t(p', \emptyset), \end{aligned}$$

where the inequality follows from Blackwell's (1953) Theorem, and the second equality follows from (19). □

Proof of Lemma 2. Fix a t, and assume by induction that (1) holds for all $s > t$.

First, consider the case where $A_s^B = 1$ for some $s < t$. By informativeness, $Pr(\tau < t | i = G) < 1$. Thus $R_{t-1} = 1$. It follows that $V_s^R(p, 1) = V_s^R(p, \emptyset) = 1$ for all $s \geq t$ since

$$V_t^D(1,1) - V_t^D(1, \emptyset) > 0 \text{ and } V_t^D(0,1) - V_t^D(0, \emptyset) < 0,$$

(1) holds in t.

Next, suppose $A_s^B < 1$ for all $s < t$. Consider the case where $A_t^B = 1$. By the informativeness property, $Pr(\tau = t | \tau \geq t, i = G) < 1$. Thus,

$$V_t^R(p, 1) < V_t^R(p, \emptyset) = 1 \text{ for all } p, \tag{20}$$

and so $V_t(0,1) < V_t(0, \emptyset)$. To show that (1) holds in t, it remains to show that $V_t(1,1) > V_t(1, \emptyset)$. Assume by contradiction $V_t(1,1) \leq V_t(1, \emptyset)$. If $t = T$, then $V_t(p_0, \emptyset) > V_t(p_0, 1)$, and thus $A_t^B = 0$. Contradiction. If $t < T$, it follows from the inductive assumption that $V_t(1, \emptyset) = V_{t+1}(1,1)$. So,

$$V_{t+1}(1,1) \leq V_t(1,1).$$

By (20) and the fact that $V_{t+1}^D(0,1) > V_t^D(0,1)$ it follows that $V_{t+1}(0,1) > V_t(0,1)$, and thus $V_{t+1}(p_0, 1) > V_t(p_0, 1)$. So, B can profitably deviate by choosing $A_t^B = 0$ and $A_{t+1}^B = 1$. Contradiction.

Next, consider the case where $A_t^B = 0$. By the informativeness property, $Pr(\tau = t|\tau \le t, i = G) > 0$. Thus, $V_t^R(p, \emptyset) < V_t^R(p, 1) = 1$ for all p, and so $V_t(1, 1) > V_t(1, \emptyset)$. To show (1) holds in t, it remains to show $V_t(0, 1) < V_t(0, \emptyset)$. Assume by contradiction $V_t(0, 1) \ge V_t(0, \emptyset)$. Since $V_t(p, 1)$ is linear in p and by Lemma 1 $V_t(p, \emptyset)$ is convex in p,

$$V_t(p, 1) > V_t(p, \emptyset) \text{ for all } p > 0.$$

It follows that

$$V_t(p_0, 1) > V_t(p_0, \emptyset) \ge V_t^B(p_0, \emptyset).$$

Thus B can profitably deviate by playing $A_t^B = 1$. Contradiction.

Finally, consider the case where $A_t^B \in (0, 1)$. I claim that $A_{t+1}^B > 0$ whenever $t < T$. Suppose not, by contradiction. By the informativeness property, $Pr(\tau = t + 1|i = G) > 0$. First, consider the case where $V_1^D(p_0, 1) = K_0(1 - p_0) + K_1 p_0 \le 0$. Because $A_t^B \in (0, 1)$, $V_t^R(p_0, 1) < 1$ and because $A_{t+1}^B = 0$, $V_{t+1}^R(p_0, 1) = 1$. Thus, $V_t(p_0, 1) < V_{t+1}(p_0, 1)$. It follows that $A_t^B = 0$ and $A_{t+1}^B = 1$ is a profitable deviation.

Next, consider the case where $V_1^D(p_0, 1) > 0$. Since $V_{t+1}^R(p_0, 1) = 1$, $V_{t+1}(p_0, 1) > V_\tau(p_0, 1)$ for all $\tau \not\le t + 1$, and thus $A_{t+1}^B = 1$ is a profitable deviation. It follows from the above statement that B is indifferent between acting at t and $t + 1$ when $t < T$. Thus,

$$V_t(p_0, 1) = V_{t+1}(p_0, 1) \tag{21}$$

Now, we want to show $V_t(1, 1) > V_t(1, \emptyset)$. Suppose not by contradiction. By the inductive assumption, $V_t(1, \emptyset) = V_{t+1}(1, 1)$ and $V_t(1, 1) \le V_{t+1}(1, 1)$. Thus, by (21), $V_t(0, 1) \ge V_{t+1}(0, 1)$. If $X < 1$ this implies

$$R(t, 1) < R(t + 1, 1) \text{ and } R(t, 0) > R(t + 1, 0),$$

which is a violation of Assumption 2. Now suppose $X = 1$. First, suppose $R(t, 1) < R(t+1, 1)$. In order for (21) to be satisfied, $R(t, 0) > R(t+1, 0)$, which again implies a violation of Assumption 2. We can analogously show that we cannot have $R(t, 0) > R(t + 1, 0)$. So, it follows that $R(t, 1) = R(t + 1, 1)$ and $R(t, 0) = R(t + 1, 0)$. When $t = T$, this implies a failure of informativeness. When $t < T$, by the inductive assumption,

$$V_t(p, \emptyset) > V_{t+1}(p, 1) = V_t(p, 1) \text{ for all } p > 0.$$

Thus, $Pr(\tau = t|i = G, \theta) = 0$ for all θ. Since $A_t^B > 0$, this implies $R(t, \theta) = 0$ for all θ, and thus $V_t(p, 1) = 0$ for all p. But by informativeness, $R(s, \theta) > 0$ for some $s > t$, θ. Thus, B can profitably deviate by playing $A_t^B = 0$. □

Proof of Lemma 4. Fix a t. Suppose by induction $p_s^* < 1$ for all $s < t$. We want to show $p_t^* < 1$. First consider the case where $A_s^B = 1$ for some $s < t$. This implies $X < 1$, otherwise, $A_s^B = 1$ cannot be part of a fixed point. It also implies $R_{t-1} = 1$, and thus $p_t^* = \hat{p}_t \in (0, 1)$. Next, consider the case where $A_s^B < 1$ for all $s < t$. First, suppose that $A_t^B = 0$. Then $R(t, 1) = 1$ for all θ, so $p_t^* \leq \hat{p}_t < 1$. Next, suppose that $A_t^B > 0$. This implies that $V_t(1, 1) > V_t(1, \emptyset)$. Thus, $p_t^* < 1$.

□

Proof of Lemma 5. Fix any fixed point x of Φ. First, I show that (x, R^x) satisfies Assumption 1 and Assumption 2. That Assumption 2 is satisfied follows from the assumption that G plays cutoff strategies in every t. To see why Assumption 1 holds, note by Lemma 4 that for all t, $p_t^* < 1$. Now fix a t and suppose that $A_s^B < 1$ for all $s < t$.

If $A_t^B = 0$, then

$$Pr(\tau = t|i = B, \theta, \tau \geq t) = 0 \text{ and } Pr(\tau = t|i = G, \theta, \tau \geq t) > 0 \text{ for all } \theta,$$

so Assumption 1 is satisfied. If $A_t^B > 0$, then

$$Pr(\tau = t|i = B, \theta = 1, \tau \geq t) = Pr(\tau = t|i = B, \theta = 0, \tau \geq t).$$

However, since G plays a cutoff strategy,

$$Pr(\tau = t|i = G, \theta = 1, \tau \geq t) > Pr(\tau = t|i = G, \theta = 0, \tau \geq t),$$

so Assumption 1 is again satisfied.

It remains to show that (x, R^x) is an equilibrium. It follows by definition that R^x is consistent with Bayes' Rule, given x. Next, I will show that given R^x, $(p_t^*)_{t=1}^T$ and $(A_t^B)_{t=1}^T$ are optimal for G and B, respectively. Since x is a fixed point, $A_t^B \in \Phi_t^B(x)$ for all x and the optimality of A_t^B follows from the definition of Φ_t^B. Next, consider G. First, we want to show that at any t, it is optimal for G to play a cutoff strategy.

Now, note that because A_t^B is optimal under this fixed point for any t, the reasoning of Lemma 2 applies here. Thus, (1) holds here. Because $V_t(p, \emptyset)$ is convex in p (Lemma 1) and $V_t(1, p) = pR(t, 1) + (1 - p)R(t, 0)$ is linear in p, playing cutoff strategy $\hat{p}_t$ is optimal. It remains to show that for all t, $\hat{p}_t = p_t^*$. Fix a t and suppose by induction that $\hat{p}_s = p_s^*$ for all $s > t$. By the definition of Φ_t^G, it follows that $\Phi_t^G(x) = p_t^* = \hat{p}_t$.

□

Proof of Lemma 6. Fix a t, and assume A^G and A^B satisfy the given assumptions. We want to show that $R_t \in (0, 1)$. Proof by induction.

Base case: $s = 0$. $R_s = R_0 \in (0, 1)$ by assumption.

Induction step: For any $s \in \{1, ..., t\}$, assume $R_{s-1} \in (0, 1)$. We want to show that $R_s \in (0, 1)$. It follows from Bayes Rule that

$$R_s = \frac{1}{1 + \frac{Pr(\tau \neq s | \tau \geq s, i=B)}{Pr(\tau \neq s | \tau \geq s\}, i=G)}}. \tag{22}$$

To show that $R_s \in (0, 1)$, it suffices to show that both the conditional probabilities in (22) lie in $(0, 1)$. In equilibrium,

$$Pr(\tau \neq s | \tau \geq s, i = B) = 1 - A_t^B \in (0, 1),$$

where $A_t^B \in (0, 1)$ holds by assumption. It remains to show that $Pr(\tau \neq s | \tau \geq s, i = G) \in (0, 1)$. To this end, because the good agent is playing a cutoff strategy,

$$H_t(p_t | p_{t-1}) = \begin{cases} 0 & \text{for all } p < p_t^* \\ \frac{G_t(p_t|p_{t-1}) - G_t(p_t^*|p_{t-1})}{G_t(p_t^*|p_{t-1})} & \text{for all } p > p_t^* \end{cases}$$

We can write

$$Pr(\tau \neq s | \tau \geq s, i = G) = \int_0^1 G_t(p_t^* | p_{t-1}) dH_{t-1}(p_{t-1}). \tag{23}$$

Now, we make two observations:

1. $G_t(p_t^* | p_{t-1}) \in (0, 1)$ for all $p_{t-1} \in (0, 1)$.

2. $H_{t-1}(p_{t-1})$ is continuous in p_{t-1}, following from the continuity of $G_{t-1}(p_{t-1}|p_{t-2})$ in p_{t-1}.

It follows from the above two observations, combined with (23) that $Pr(\tau \neq s|\tau \geq s, i = G) \in (0, 1)$.

$\square$

Proof of Lemma 7. Fix any $p \in (0, 1)$. It suffices to show that for each t, there exists $\overline{p}_t \in (0, 1)$ such that if $p_0 < \overline{p}_t$, then $p^*_t < p$ for any equilibrium under p_0.

Suppose not, by contradiction. Let s be the last period such that the statement fails. Since by Proposition 3 $p^*_T = p_0$ in any equilibrium, $s < T$. Thus, there exists a $q > 0$ and a sequence of priors $(p^n_0)^\infty_{n=1}$ such that $\lim_{n\to} p^n_0 = 0$, where

$$p^{*,n}_s > q \text{ for all } n \text{ and } \lim_{n\to\infty} p^{*,n}_{s+1} = 0, \tag{24}$$

where $p^{*,n}_t$ is some equilibrium under p^n_0. Let superscript n refer to objects under this equilibrium.

It follows from $B's$ indifference (Proposition 3) that

$$\lim_{n\to\infty} [R^n(s+1, \theta = 0) - R^n(s, \theta = 0)] = 0. \tag{25}$$

Thus, in order for the $p^{*,n}_s$ to be optimal for all n

$$\lim_{n\to\infty} [R^n(s+1, \theta = 1) - R^n(s, \theta = 1)] = 0. \tag{26}$$

Now, it follows from Bayes Rule that for any t,

$$R^n(t, \theta) = \frac{1}{1 + \frac{1-R^n_{t-1}}{R^n_{t-1}} X^n(t, \theta)},$$

where

$$X^n(t, \theta) \equiv \frac{A^{B,n}_t p^n_0}{Pr(\tau = t|\tau \nless t, i = G) Pr(\theta|\tau = t, i = G)}.$$

Thus, in order for (25) and (26) to hold,

$$\lim_{n\to\infty} X^n(s, \theta) = \lim_{n\to\infty} X^n(s+1, \theta) \text{ for } \theta \in \{0, 1\}. \tag{27}$$

However, it follows from (24) that

$$\lim_{n\to\infty}\frac{Pr^n(\theta=1|\tau=s,i=G)}{Pr^n(\theta=1|\tau=s+1,i=G)}=\infty \text{ while } \lim_{n\to\infty}\frac{Pr^n(\theta=0|\tau=s,i=G)}{Pr^n(\theta=0|\tau=s+1,i=G)}=0,$$

which implies that (27) fails. Contradiction.

□

Proof of Lemma 8. I begin by showing $A_t^B < 1$ for any t. To this end, I will prove the stronger statement that there exists $\underline{X} \in (0,1)$ and $\underline{b} \in (0,1)$ such that $A_t^B < \underline{b}$ in any equilibrium under any β when $X > \underline{X}$.

Proof by induction. Fix any t. Suppose there exists an $\underline{X}_{t-1}$ and$\underline{b}_{t-1} \in (0,1)$ such that $A_s^B < \underline{b}_{t-1}$ for all $s < t$ when $X > \underline{X}_{t-1}$. This holds vacuously when $t = 1$. We want to show that there exists $\underline{X}_t \in (\underline{X}_{t-1}, 1)$ and $\underline{b}_t$ such that $A_s^B < \underline{b}_t$ for all $s \le t$ when $X > \underline{X}_t$. Suppose not, by contradiction. Then, there exists a sequence of X, $(X_n)_{n=1}^{\infty}$ with $X_n \to 1$ where $A_t^{B,n} \to 1$ for some equilibrium strategy $A_t^{B,n}$ for B under some value of β under X_n. Let superscript n denote equilibrium objects under each such equilibrium. I claim $V_t^{B,n}$ must satisfy the following two properties:

1. There exists an N and $\underline{V} < 1$ such that $V_t^{B,n}(p_0, 1) < \underline{V}$ for all $n > N$.
2. $\lim_{n\to\infty} V_t^{B,n}(p_0, \emptyset) = 1$.

These two properties imply that there exists an N such that for all $n > N$, $V_t^{B,n}(p_0, 1) < V_t^{B,n}(p_0, \emptyset)$, thus implying that $A_t^{B,n} = 0$, a contradiction.

I begin by establishing 1. Since $X_n \to 1$, it suffices to show that there exists N and $\overline{R} < 1$ such that $R^n(t, 1) < \overline{R}$ for $n > N$. The inductive assumption implies there exists $\hat{R} < 1$ such that $R_{t-1}^n < \hat{R}$. Since $A_t^n \to 1$, there exists an N and $\underline{A} > 0$ such that for all $n > N$, $A_t^n > \underline{A}$. Recall that

$$R^n(t,1) = \frac{1}{1 + \frac{1-R_{t-1}}{R_{t-1}} \frac{p_0 A_t^{n,B}}{Pr^n(\tau=t|i=G,\tau\ge t,\theta=1)}}.$$

It thus follows that for all $n > N$,

$$R^n(t,1) < \overline{R} \equiv \frac{1}{1 + \frac{1-\hat{R}}{\hat{R}} \underline{A} p_0} < 1.$$

Next, I establish 2. It suffices to show $R_t^n \to 1$. We know by assumption that $A_t^{B,n} \to 1$. Note that for any n such that $A_t^{B,n} = 1$, by the informativeness assumption, $p_t^{*,n} < 1$, and thus $R_t^n = 1$. Let X_m denote the subsequence of $(X_n)_{n=1}^{\infty}$ such that $A_t^n < 1$. Note that if X_m is finite, then the statement holds trivially. Now suppose X_m is an infinite subsequence. $A_t^m < 1$ implies $V_t^{B,m}(p_0, 1) \leq V_t^{B,m}(p_0, \emptyset)$. This implies $V_t^{G,m}(p_0, 1) \leq V_t^{G,m}(p_0, \emptyset)$ and thus $p_t^{*,m} \geq p_0 > 0$. This then implies that there exists $q > 0$ such that $Pr^m(\tau \neq t | \tau \geq t, G) > q$. Furthermore, by the inductive assumption, $p_{s,m}^* > p_0$ for all m and $s < t$. So ,there exists a $\hat{R} > 0$ such that $R_{t-1}^m > \hat{R}$ for all m. Thus,

$$\lim_{m\to\infty} R_t^m = \lim_{m\to\infty} \frac{1}{1 + \frac{1-R_{t-1}^m}{R_{t-1}^m} \frac{1-A_t^m}{Pr^m(\tau \neq t | \tau \geq t, i=G)}} = 1.$$

Now, I show that $A_t^B > 0$ for any t. Suppose not, by contradiction. Then there exists a sequence of X $(X_n)_{n=1}^{\infty}$ with $X_n \to 1$ where $A_t^{B,n} = 0$ for all n under some equilibrium strategy $A_t^{B,n}$ under X_n and some β. Since G acts with positive probability in equilibrium under any t by Lemma 2, $R^n(t, \theta) = 1$ for all θ. Since $X_n \to 1$, $\lim_{n\to\infty} V_t^{B,n}(p_0, 1) = 1$. Thus, $V_1^{B,R,n}(p_0) \to 1$. Since $R_0 < 1$, this implies that for n sufficiently large, B's expected reputation at the end of the game exceeds R_0. Thus R^n cannot be consistent with Bayes Rule. Contradiction. □

Proof of Lemma 9. I begin by establishing the bound $\underline{R}$. by Lemma 8, in every equilibrium under any β and t, $V_t^B(p_0) = V_t^B(p_0, \emptyset)$. This implies $V_t^G(p_0, 1) \leq V_t^G(p_0, \emptyset)$, and thus $p_t^* \geq p_0$ for all t. Thus there exists $\underline{p} \in (0, 1)$ such that $Pr(\tau = t | \theta = 0, i = G) \geq \underline{p}$ in any equilibrium, under any β. Namely, one such $\underline{p}$ is the probability $\tau = t$ given $\theta = 0$, $i = G$ and the good agent plays cutoff strategies $p_s^* = p_0$ for all $s < t$ and $p_t^* = \overline{p}_t$. Thus, in any equilibrium

$$R(t, 0) \geq \underline{R} \equiv \frac{1}{1 + \frac{1-R_0}{R_0} \frac{1}{\underline{p}}} > 0.$$

Now, I establish the bound $\overline{R}$. Suppose by contradiction that there does not exist such a $\overline{R}$. Then, there exists a sequence of equilibria, indexed by n (under possibly different values of β) such that

$$\lim_{n\to\infty} R^n(t,1) = 1. \tag{28}$$

Now, recall that

$$R^n(t,1) = \frac{1}{1 + \frac{1-R_0}{R_0}\frac{Pr^n(\tau=t|i=B,\theta=1)}{Pr^n(\tau=t|i=G,\theta=1)}}.$$

Thus, in order for (28) to hold, $\lim_{n\to\infty} Pr^n(\tau = t|i = B, \theta = 1) = 0$. Since $Pr^n(\tau = t|i = B, \theta = 0) = \frac{1-p_0}{p_0} Pr^n(\tau = t|i = B, \theta = 1)$,

$$\lim_{n\to\infty} Pr^n(\tau = t|i = B, \theta = 0) = 0.$$

However, as previously shown for all n, there exists $\underline{p} > 0$ such that

$$Pr^n(\tau = t|\theta = 0, i = G) \geq \underline{p}.$$

This implies

$$\lim_{n\to\infty} \frac{Pr^n(\tau = t|\theta = 0, i = B)}{Pr^n(\tau = t|\theta = 0, i = G)} = 0,$$

and therefore $\lim_{n\to\infty} R^n(t,0) = 1$. Since $\lim_{n\to\infty} R^n(t,\theta) = 1$ for all θ, it follows that for n sufficiently large, B can profitably deviate by playing the pure strategy $A_t^{B,n} = 1$. Contradiction. □

Proof of Proposition 5. First, we want to show that both players play cutoff strategies in every t. Note that the proofs of Lemma 1 and Lemma 2 do not rely on the assumption that B is unable to learn, and thus apply in this setting as well. Now note that Lemma 1 applies to B's value function as well, i.e., $V_t^B(p, \emptyset)$ is convex in p. Next, because $V_t^G(\theta, a) = V_t^B(\theta, a)$ for all $\theta \in \{0, 1\}$ and $a \in \{\emptyset, 1\}$, Lemma 2 applies to B's value function as well, i.e.,

$$V_t^B(1,1) > V_t^B(1,\emptyset) \text{ and } V_t^B(0,\emptyset) > V_t^B(0,1) \text{ for all } t.$$

That $p_T^{*,B} = p_T^{*,G}$ follows immediately from the fact that $V_T^B(p,a) = V_T^G(p,a)$ for all p, a. To show $p_t^{*,B} < p_t^{*,G}$ for all $t < T$, note $V_t^B(p,1) = V_t^G(p,1)$ for all t, but because G's signal Blackwell dominates B's signal and has a strictly larger support on the likelihood ratios:

$$V_t^B(p,\emptyset) < V_t^G(p,\emptyset) \text{ for all } t < T \text{ and } p \in (0,1).$$

The statement follows immediately. □

Proof of Proposition 6. We wish to extend the proof of Theorem 1 to the setting where the bad agent observes signals drawn from f^B. Thus, fix an f^B with support $(\underline{z}, \overline{z})$, where $\underline{z} > 0$ and $\overline{z} < \infty$. First, note that Lemma 3 extends immediately. Now, rather than letting A_t^B refer to the bad agent's strategy in t, given B's strategy $(p_t^{*,B})_{t=1}^T$, let

$$A_t^B \equiv Pr(\tau = t | \tau \geq t, i = B).$$

Let us begin by extending the proof of existence of $\overline{K}_1$ (again, the existence of $\overline{K}_0$ follows analogously). Following the proof of Theorem 1, there exists N and $\overline{W} < 0$ such that for all $n > N$, $W_s^{D,B,n}(\overline{p}_s) < \overline{W}$, where $\overline{p}_s < 1$ is the upper bound on the support of beliefs for the bad agent in s. Since $A_s^{B,n} \in (0,1)$,

$$W_s^{R,B,n}(\overline{p}_s) > -\overline{W} > 0.$$

So it again follows that for all $n > N$,

$$R^n(s,1) - R^n(s+1,1) > -\overline{W},$$

a contradiction.

Next, we will extend the proof of the existence of $\overline{p}$. We will extend each step of the proof in turn. Again, we assume by contradiction that there exists a sequence $\{p_{0,n}\}_{n=1}^{\infty}$ where $p_{0,n} \in (0,1)$ and $\lim_{n\to\infty} p_{0,n} = 0$, such that for all n $p_{t,n}^* \geq \hat{p}_t$. Hereafter, let $\overline{p}_{t,n}$ denote the upper bound on the support of beliefs that the bad agent might hold at t under prior $p_{0,n}$.

To extend step 1, we will instead show that there exists an N such that for all $n > N$, $A_{s,n}^B > 0$ for all $s \geq t$. The proof that $A_{t,n}^B > 0$ for all n follows identically. Next, consider $s > t$. First note $\lim_{n\to\infty} \overline{p}_{t,n} = 0$. Thus there exists N such that for all $n > N$,

$$\overline{p}_{t,n} K_1 + (1 - \overline{p}_{t,n}) K_0 < 0.$$

Now, suppose by contradiction that for some $n > N$, $A^B_{s,n} = 0$. Then we have that

$$V^B_{s,n}(\overline{p}_{t,n}, 1) > V^B_{t,n}(\overline{p}_{t,n}, 1),$$

which contradicts that $A^B_{t,n} > 0$. Hereafter, I redefine $\{p_{0,n}\}_{n=1}^{\infty}$ to be the sequence $\{p_{0,n}\}_{n=N+1}^{\infty}$.

To extend step 2, we note by step 1 that $A^B_{s,n} \in (0,1)$ for all $s \geq t$ (that $A^B_{s,n} < 1$ is immediate). Thus for all $s \geq t$ and n, there exists $p_{s,n} \in (\underline{p}_{T,n}, \overline{p}_{T,n})$ such that

$$V^B_{s,n}(p_{s,n}, 1) = V^B_{s,n}(p_{s,n}, 1).$$

Since $\lim_{n\to\infty} \underline{p}_{T,n} = \lim_{n\to\infty} \overline{p}_{T,n} = 0$,

$$\lim_{n\to\infty} V^B_{s,n}(p_{s,n}, 1) - V^B_{s+1,n}(p_{s,n}, 1) = (1-X)(\beta^s - \beta^{s+1})K_0 + X \lim_{n\to\infty} [R^n(s,0) - R^n(s+1,0)] = 0.$$

Thus step 2 follows from the same reasoning as in Theorem 1.

To extend step 3, first define

$$Q^B_{n,t} \equiv \frac{\int_0^1 \int_{p^{*,B}_{t,n}}^1 (1-p_t) dG^B_{t,n}(p_t|p_{t-1}) dH^B_{t,n}(p_{t-1})}{\int_0^1 \int_{p^{*,B}_{t,n}}^1 p_t dG^B_{t,n}(p_t|p_{t-1}) dH^B_{t,n}(p_{t-1})},$$

where $G^B_{t,n}$ and $H^B_{t,n}$ are defined analogously for the bad agent as they are for the good agent. We want to show

$$\lim_{n\to\infty} \frac{Q_{n,t}}{Q^B_{n,t}} = 0.$$

As before, $Q_{n,t} < \frac{1}{p^*_{t,n}} - 1$. Meanwhile, since $p^{*,B}_{t,n} < \overline{p}_{t,n}$ for all n,

$$Q^B_{n,t} > \frac{\int_0^1 \int_{p^{*,B}_{t,n}}^1 dG^B_{t,n}(p_t|p_{t-1}) dH^B_{t,n}(p_{t-1})}{\int_0^1 \int_{p^{*,B}_{t,n}}^1 \overline{p}_{t,n} dG^B_{t,n}(p_t|p_{t-1}) dH^B_{t,n}(p_{t-1})} - 1 = \frac{1}{p_{t,n}} - 1.$$

Recall that $\lim_{n\to\infty} \overline{p}_{t,n} = 0$. The statement follows immediately. It follows from step 2. that

$$\frac{\int_0^1 \int_{p_{t,n}^{*,B}}^1 (1-p_t) dG_{t,n}^B(p_t|p_{t-1}) dH_{t,n}^B(p_{t-1})}{\int_0^1 \int_{p_{t,n}^{*}}^1 (1-p_t) dG_{t,n}(p_t|p_{t-1}) dH_{t,n}(p_{t-1})} \text{ is bounded.}$$

Thus,

$$\lim_{n\to\infty} \frac{\int_0^1 \int_{p_{t,n}^{*,B}}^1 p_t dG_{t,n}^B(p_t|p_{t-1}) dH_{t,n}^B(p_{t-1})}{\int_0^1 \int_{p_{t,n}^{*}}^1 p_t dG_{t,n}(p_t|p_{t-1}) dH_{t,n}(p_{t-1})} = \lim_{n\to\infty} \frac{\int_0^1 \int_{p_{t,n}^{*,B}}^1 (1-p_t) dG_{t,n}^B(p_t|p_{t-1}) dH_{t,n}^B(p_{t-1})}{\int_0^1 \int_{p_{t,n}^{*}}^1 (1-p_t) dG_{t,n}(p_t|p_{t-1}) dH_{t,n}(p_{t-1})} \frac{Q_{n,t}}{Q_{n,t}^B} = 0.$$

So, $\lim_{n\to\infty} R_n(t,1) = 1$.

The extension of step 4 is immediate. A contradiction follows as in the proof of Theorem 1.

□

Proof of Proposition 7. First, I argue Lemma 2 must hold when $T=\infty$ and $X=1$. I now show $V_t(1,1) > V_t(1,\emptyset)$ (that $V_t(0,1) < V_t(0,\emptyset)$ follows analogously). Suppose by contradiction $V_t(1,1) \leq V_t(1,\emptyset)$. This implies $R(t,1) \leq R(\tau,1)$ where $\tau \equiv \arg\max_{\emptyset \cup s>t} R(s,\theta=1)$. If $R(t,0) \leq R(\tau,0)$ this implies that either G or B could profitably deviate (in G's case, by never acting at t). If $R(t,0) > R(\tau,0)$, this implies that either G or B could profitably deviate, or in the case where $R(t,1) < R(\tau,1)$, there is a failure of Assumption 2.

That Lemma 2 holds implies that $R(t,\theta=1)$ ($R(t,\theta=0)$) is strictly decreasing (increasing) in t. Because they are also bounded, by the Monotone Convergence Theorem, there exists R^1 and $R^0 \in [0,1]$ such that

$$\lim_{t\to\infty} R(t,1) = R^1 \text{ and } \lim_{t\to\infty} R(t,0) = R^0. \tag{29}$$

This implies that there must exist a t such that $V_t(\emptyset,p)$ is convex, and thus the proof of Lemma 1 must extend to this setting. It follows that G plays a cutoff strategy $p_t^* \in (0,1)$ and B mixes in every t. Now note that because of this

$$R(\emptyset,1) < R(\emptyset,0) \text{ and } R(t,1) > R(t,0) \text{ for all t.}$$

Thus, by (29), $\lim_{t\to\infty} p_t^* = 1$. In order for R to be consistent with Bayes Rule, this implies that one of the two limits must hold:

$$\lim_{t\to\infty} R(t,1) = 1 \qquad \lim_{t\to\infty} R(t,0) = 0.$$

However, this contradicts the fact that $R(t,1)$ is strictly decreasing, and $R(t,0)$ is strictly increasing in t. □

Proof of Corollary 4. Fix any $T < \infty$. Restrict attention to strategies where B and G never act after T. Furthermore, let $R(t,\theta) = 0$ for all $t > T$, θ. For all $t \leq T$ let the strategies and R be those specified by an equilibrium under a deadline T, the existence of which is guaranteed by Proposition 1. It follows that this is a T-informative equilibrium. □