Competition and Herding in Breaking News

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Abstract

I present a dynamic model of breaking news. News firms are rewarded for reporting before their competitors but also for making reports that are credible to consumers. Errors occur when firms *fake*, reporting a story despite lacking evidence. While errors occur in equilibrium even under a monopoly, competition and observational learning exacerbate errors and give rise to rich dynamics in firm behavior. Competition intensifies faking by engendering a preemptive motive, but the haste-inducing effect of preemption is endogenously mitigated by gradual improvement in report credibility over the course of a news cycle. Meanwhile, observational learning causes existing errors to propagate through the market. This is driven by a *copycat effect*, in which one report triggers an immediate surge in faking by others. This behavior is consistent with herding on the decision to report a story as well as herding on the timing of reports.

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1. Introduction

What a newspaper needs in its news, in its headlines, and on its editorial page is terseness, humor, descriptive power, satire, originality, good literary style, clever condensation, and accuracy, accuracy, accuracy!

— Joseph Pulitzer

Accuracy is often considered to be the core tenet of news media. This belief is widely held by consumers of news: when asked in a 2018 Pew survey, the majority of respondents listed accuracy as a primary function of news, valuing it over thorough coverage, unbiasedness, and relevance.

Despite this, public perceptions of news accuracy are not favorable. In a 2020 survey, 38% of respondents stated that they go into a news story thinking it will be largely inaccurate. While many factors may contribute to this skepticism, consumers express particular concern about hasty reporting: 53% of respondents state that news breaking too quickly is a major source of errors.

These concerns are supported by a multitude of instances in which news media have made major factual errors. In the immediate aftermath of the 9/11 attacks, cable news stations made several statements that were false: NBC reported an explosion outside the pentagon, CNN reported a fire outside the national mall, and CBS claimed the existence of a car bomb outside the state department. Erroneous reporting has been endemic to terrorist attacks in general, with news media misidentifying perpetrators or other key details of the Boston bombings, Sandy Hook massacre, London bombings, and Oklahoma City bombings. Furthermore, such errors are not limited to terrorist attacks. In 2004, CBS news, under the direction of Dan Rather, published the Killian documents, a collection of memos which called into question George W. Bush's military record. These documents could never be authenticated and were widely believed to be forged. More recent media blunders are ever present: in 2017, ABC news falsely reported that Michael Flynn would testify that Donald Trump had directed him "to make contact with the Russians." In 2019, ABC News headlined its nightly news broadcast with what it claimed to be exclusive footage of the ongoing air strikes on Syria. It was later revealed that this footage was from a machine gun convention in Oklahoma.

While such errors are commonplace, they are also costly to news firms. For one, exposure of errors can be reputationally damaging. This was especially true of the *Rolling Stone* scandal, in which the magazine falsely accused a group of University of Virginia students of sexual assault. Not only was the journalistic failure widely reported by other firms, the error resulted in several publicized lawsuits against the magazine. Furthermore, such errors often lead firms to oust journalists in an apparent effort to protect their reputations. This was evident in the terminations of Dan Rather and Brian Ross —both lead journalists at major news stations—following their respective reporting blunders.

The objective of this paper is twofold. First, I seek to understand why reporting errors are pervasive despite their costliness to firms. In particular, I explore how strategic forces can induce firms to commit errors that are completely avoidable. My second objective is to understand *when* reporting errors are most probable, and relatedly, when firms are less trustworthy. That is, I seek to understand both the dynamics of reporting errors and the environmental factors that can make them more prevalent.

Model To answer these questions, I present a dynamic model of breaking news. I consider a continuous-time setting where multiple firms dynamically and privately learn about a story and choose if and when to report it. Firms learn by seeking confirmation that the story is true. Reporting errors occur when firms *fake*, i.e., report the story despite lacking confirmation. Because reports are public, firms also learn by observing the reports of their competitors. I thus account for an important feature of the newsroom: firms learn privately but also observationally.

In this model, firms seek viewership. Error-prone reporting conflicts with this objective, and is thus costly, in two ways. First, errors harm firms ex post (after they have been exposed). This captures the negative effect of errors on a firm's future livelihood. Error-prone reporting is also costly ex ante (before errors can be unearthed). This is because a viewership hinges on *credibility*, i.e., the consumer's belief that the report is not fake. I thus take the stance that a story is valued to the extent that there is trust in a firm's journalistic standards, a notion that is informed by consumers' stated preference for accurate news. Finally, this model accounts for a salient feature of the breaking news problem: preemption. All else equal, a firm who preempts its rivals (e.g., by being the first to report) is rewarded with greater viewership.

Analysis I establish both existence and uniqueness of an equilibrium. Under this equilibrium, fake reports never occur with point mass, but are rather generated according to a non-homogenous Poisson process. This behavior implies an indifference condition: at any time in which the firm might fake, it must be indifferent between faking immediately and after some short wait. This condition in turn implies an ordinary differential equation

(ODE) on the arrival rate of fake reports. I show that the equilibrium is characterized by a recursive system of such ODEs, a result which is central to the analysis and guides the economic implications that follow.

Economic Implications I find that errors are strategic responses to three features of the newsroom setting: a lack of commitment by firms, competition, and observational learning. Competition and observational learning not only exacerbate faking, but also introduce distinct dynamics in firm behavior.

I begin by showing that errors can occur even in the absence of competition. In particular, if the ex-post cost of error is relatively small —because consumers are less aware or critical of them —even a monopolist will fake. I argue that such errors are driven by a firm's inability to commit to a reporting strategy. Because consumers cannot detect faking, the market share a firm enjoys from a report depends only on consumers' beliefs about its quality. The firm is thus tempted to fake after these beliefs have been formed. I substantiate this intuition by showing that a firm who could commit would always report truthfully, and thus never err.

I then analyze a setting where multiple firms report simultaneously. I find that both competition and observational learning can exacerbate errors, but do so in two different ways. Competition can give rise to a preemptive motive in equilibrium: firms have an incentive to speed up their reporting in order to beat out competitors. This incentive for speed induces firms to fake. To restore the firm's indifference condition, credibility must fall to a sufficiently low level. In equilibrium, such lower credibility can only be consistent with higher faking. Notably, this preemptive motive is not purely an artifact of the firm's payoff function. Rather, it is a feature of the equilibrium which only prevails under certain parameter values. In fact, I show that this preemptive motive will endogeously disappear whenever the ex-post cost of error is sufficiently small. Meanwhile, observational learning exacerbates errors by causing existing errors to propagate through the market. When one firm reports a story, other firms become more confident that the story is true. This increased confidence will yield succeeding firms more inclined to fake. The effect of observational learning is especially salient when combined with a preemptive motive: in such scenarios, observational learning can cause faking to increase so much that it causes a deterioration in credibility.

This paper also sheds light on the dynamics of reporting behavior and credibility. These dynamics take two different forms in equilibrium: gradual changes that happen in the absence of new reports and discrete changes that occur in response to a new report. I first show that firms become gradually more truthful —i.e., less inclined to fake—conditional on no new reports being made. Furthermore, firms become more credible over time whenever preemptive concerns are present. In other words, consumers are less trusting of reports that are made quickly. This finding aligns with consumers' expressed concerns about hasty reporting. The reason for this gradual improvement in credibility lies in the firms incentives. The risk of being preempted introduces an endogenous cost to delay. The firm must somehow be compensated for this cost to ensure that its indifference condition is satisfied. This is achieved by means of increasing credibility. That is, increasing credibility mitigates the haste-inducing effects of preemption enough to yield the firm indifferent between faking and truth telling.

Firms also exhibit discrete changes in reporting behavior in response to a rival report. This can entail a *copycat effect*, in which one firm's report causes an instantaneous boost in faking by others. The copycat effect implies that when one firm's report is quickly corroborated by other firms, such follow-up reports will often lack credibility because they are not independently verified. It suggests that firms herd on both the reports themselves, both erroneous and valid ones, and the timing of such reports. In addition to anecdotal evidence of clustering in the timing of news errors [1], such herding has been documented in the empirical literature. Notably, Cagé, Hervé, and Viaud (2020) find that in 25% of cases, a news story is reported by a different media outlet within 4 minutes of being published by the original news breaker. I thus provide rationale for such herding that is grounded in both the strategic and learning environment news media face.

In addition to these core results, I consider comparative statics and an extension to a heterogeneous setting. I find that both a higher ex-post cost of error and a higher learning ability improve credibility in equilibrium. I further explore the role of competition by considering the marginal effect of an additional firm in the market. Whenever preemptive concerns are present, adding a competitor makes each individual firm more likely to fake early on by increasing the preemptive threat they face. However, this is mitigated later on by the effects of observational learning: existing firms are able to learn that the story is false more quickly by observing the silence of an additional competitor, which will yield them less willing to fake. Finally, I extend the model to allow for heterogeneity in firms' ability to learn, finding that firms with greater ability to learn are also more credible in equilibrium.

This paper also sheds light on the consequences of media mergers. As noted in Anderson and McLaren (2012), media mergers could have consequences for consumers beyond those in conventional markets. They argue that the merging of news firms could give rise

[1] Examples include the reporting errors surrounding the Boston bombings and the 2000 US presidential election.

to media bias. Meanwhile, I demonstrate that it can also impact news accuracy: to the extent that mergers do not ward off entrants into the market, they can weaken a firm's motive to make errors. Thus, in contrast to Anderson and McLaren (2012), I show that media mergers may have an unforeseen positive effect on news quality.

Related Literature The preemption literature has modeled a variety of scenarios, including R&D races (Fudenberg, Gilbert, Stiglitz, and Tirole (1983)), technology adoption (Fudenberg and Tirole (1985)), the strategic exercise of options (Grenadier (1996)), and financial bubbles (Abreu and Brunnermeier (2003)). This paper contributes to this literature in two ways. The first is in the endogeneity of the payoff function. In the existing literature, a player's decision to preempt does not affect its underlying payoff. That is, the benefit of preempting is exogenous. I consider setting where a firm's payoff from reporting hinges on the consumer's beliefs about the quality of its report. Such beliefs are important in the market for news because consumers may not be able to immediately observe the quality of a news report, e.g., whether it was verified before being reported. This assumption has implications for the nature of the firm's incentives. While in the existing literature, players earn some exogenous benefit from delaying their actions which counteracts the incentive to preempt, this does not hold in our setting. Rather, I find that even if no such benefit exists exogenously, it will arise in equlibrium.

This paper is not the first to consider observational learning in a preemption setting. In Hopenhayn and Squintani (2011), firms can only observe their own payoffs, and thus draw inferences about the payoffs of their competitors by observing when and whether they act. Meanwhile, in Bobtcheff, Bolte, and Mariotti (2017), players receive breakthroughs which are privately observed, and thus at every moment are uncertain about how much competition they face. In contrast, I assume that firms learn observationally about their own payoffs. This form of observational learning is also present in Bobtcheff, Levy, and Mariotti (2022), in which players privately observe bad-news Poisson signals in a winner-takes-all setting. Like this paper, the lack of action by one's opponent implies that acting is unprofitable. However, in contrast to Bobtcheff et al. (2022), I consider a payoff environment that is not necessarily winner-takes-all (i.e., nth movers may enjoy positive payoffs). It is precisely because of this more general payoff structure that observational learning gives rise to herding on both actions themselves and the timing of these actions. In this sense, this paper also connects to the literature on herding with endogenously-timed decisions (Gul and Lundholm (1995), Chamley and Gale (1994), Levin and Peck (2008)). In particular, the notion that an action by one individual can trigger others to quickly follow suit arises in Gul and Lundholm (1995). While such behavior is efficient in their setting, that is not the case in ours, where it can cause errors to propagate through the market.

This paper contributes to a recent literature on preemption in news (Lin (2014), Pant and Trombetta (2019), Andreottola, de Moragas, et al. (2020)). As in this paper, Lin (2014) considers a setting where firms dynamically learn about a single story and must decide whether and when to report it. Meanwhile, Andreottola et al. (2020) suppose that firms report on a sequence of potential political scandals, finding that competition improves the quality of information but deteriorates consumer welfare. I contribute to this literature by modeling two vital features of the breaking news environment, i.e., the role of credibility and observational learning. These two features are what drive many of the economic implications of this paper, including the importance of commitment, herding, and dynamics in firm behavior.

This paper also contributes more broadly to a literature on competition in news. This literature is surveyed by Gentzkow and Shapiro (2008), with more recent contributions by Liang, Mu, and Syrgkanis (2021), Galperti and Trevino (2020), Chen and Suen (2019), and Perego and Yuksel (2018). Chen and Suen (2019) and Galperti and Trevino (2020) consider the effects of competition on news accuracy. In both papers, firms compete for the attention of consumers and face constraints or costs to accuracy. In my setting, accuracy is not intrinsically costly. Rather, accurate reporting entails an indirect cost, namely that of being preempted. I contribute more generally to this literature in two ways. First, I consider the effects of competition on a different notion of accuracy, namely the prevalence of factual errors. Second, because I consider a dynamic environment, I can study the effect of competition on not only on news quality as a whole, but also on its dynamics.

Finally, this paper connects broadly to the literature on the strategic provision of information. Unlike frameworks where a sender seeks to induce a particular action from receivers (Crawford and Sobel (1982), Kamenica and Gentzkow (2011)), firms in my model treat information as a good, aiming to maximize market share. This notion underlies the literature on demand-driven media bias. In Mullainathan and Shleifer (2005), firms bias their reports in an appeal to consumers' preferences for having their beliefs confirmed. Meanwhile, in Gentzkow and Shapiro (2006) bias arises purely in response to reputational concerns, and is thus driven by an aim for long-term profitability. My framework accounts for both the short-term and long-term objectives of a news firm. This sheds light on an intertemporal tradeoff faced by news media: low-quality reporting may benefit a firm in the short run, but can cause damage in the long run. Separately, I note that the kind of deception firms engage in shares common threads with other work. The notion of faking is also studied in Boleslavsky and Taylor (2020) in a competition-free setting that encorporates discounting. Furthermore, the endogenous Poisson arrival of inaccurate information also arises in Che and Hörner (2018), and consists of spamming by recommender systems.

Outline The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 is dedicated to characterizing its equilibrium, first considering the monopoly benchmark and then encorporating competition. In Section 4, I present the core economic implications of this equilibrium, which pertain to the effects of competition and equilibrium dynamics. In Section 5, I present comparative statics. Section 6 considers an extension of the model in which firms have heterogenous learning abilities. Finally, Section 7 concludes. All formal proofs are relegated to the Appendix.

2. A model of breaking news

Here, I present the baseline model and discuss the key modeling assumptions. An extension where firms have heterogeneous learning abilities is presented in the Appendix.

There are $N \geq 1$ firms, indexed by i, and one consumer. Time, which is continuous and has an infinite horizon, is denote by $t \in [0, \infty)$. There is a time-invariant state $\theta \in \{0, 1\}$, which denotes whether a particular story is true ($\theta = 1$) or false ($\theta = 0$). At $t = 0$, all players are endowed with a common prior $p_0 \equiv Pr(\theta = 1) \in (0, 1)$.

Learning and reporting Starting from $t = 0$, firms privately learn about the state. They do so by means of a one-sided Poisson signal: if $\theta = 1$, a signal revealing that $\theta = 1$ arrives to each firm at a Poisson rate $\lambda > 0$. To formalize this, let $s_i \in [0, \infty]$ denote the time at which such a conclusive signal arrives to firm i, with $s_i = \infty$ denoting that a signal never arrives. Under this learning process, $s_i \sim (1 - e^{-\lambda s_i})$ if $\theta = 1$, and $s_i = \infty$ if $\theta = 0$. I further assume that conditional on $\theta = 1$, s_i is i.i.d. across firms. This learning process serves as an approximation for how news firms learn about a breaking news event. Given the short lifecycle of a breaking news story, one can imagine that firms do not seek piecemeal evidence but rather pursue reliable sources who can confirm the story. For instance, in the case a terrorist attack, this would entail reaching out to contacts in the police department.

In addition to learning about the story, firms report about it. Each firm has a single opportunity to make a report over the course of the game. Firms do not choose what to report, but instead whether and when to do so. As the payoff function will soon illustrate, the content of this report can be interpreted as an assertion that the story is true, i.e., that $\theta = 1$. A report history H is a partially ordered set of pairs (i, t_i), pairing each firm i who has reported with a report time t_i, with elements ordered according to the order in which the reports were made.[2] Report histories are public: at every time t, all players observe the current report history. Thus, firms learn about θ not only privately, but also observationally by means of rival firms' reports.

Payoffs A firm who never reports earns a payoff of 0. A firm who does report earns

$$k_n\alpha - \beta\mathbb{I}[\theta = 0]. \tag{1}$$

The first term ($k_n\alpha$) represents the immediate market share (i.e., viewership or readership) that the firm enjoys from reporting a story. The parameters k_n capture the role of the firm's order n, while α denotes the credibility of the firm's report. Formally, an index $n \in \{1, 2, ..., N\}$ denotes that the firm was the nth to report. The k_n are constants, where $k_1 \geq k_2 \geq ... \geq k_N \geq 0$. That is, all else equal, firms who report early compared to their competitors enjoy greater market share. The firm's payoff is also a function of the credibility (α) of its report. Credibility is the consumer's belief, at the time the report is made, that the firm has received conclusive evidence. Formally, it is the belief that $s_i \in [0, t]$, where t is the time of the firm's report. While the k_n are exogenous, α is endogenous. In assuming a product form for the market share, I take the stance that a report is profitable insofar that consumers believe it was informed. This captures the notion that consumers value accuracy in journalism, and thus only consume news to the extent that they find it credible. One can show that this formula for market share can be microfounded by assuming a continuum of consumers who (1) have a preference for accurate news stories (2) face heterogeneous costs from consuming a story and (3) multi-home across news firms. This microfoundation is presented formally in the Appendix.

The second term of (1), $-\beta\mathbb{I}[\theta = 0]$, captures the ex-post penalty of error: a firm who reports when $\theta = 0$ incurs a penalty $\beta > 0$. The constant β captures the expected reputational harm a firm suffers from making a report that is later uncovered to be false. Namely, β captures both how aware and critical consumers are of news errors.

Equilibrium A Markov strategy F is a set of distributions $F_{p,n}$ over future report times for each belief $p \equiv Pr(\theta = 1)$ and order n of the next firm to report.[3] Specifically, the span of time the firm waits before reporting, conditional on not receiving a conclusive signal,

[2] Formally, elements are ordered according to relation $\succsim$, where $(i, t_i) \succ (j, t_j)$ if $t_i > t_j$ or $t_i = t_j$ but i reported first, and $(i, t_i) \sim (j, t_j)$ indicates that the reports were made simultaneously.

[3] Formally, $n = |H| + 1$, where H denotes the current history. I assume that if m firms report at the same history H, one firm will be assigned order n, another $n + 1$, etc., with their identities randomly determined according to a uniform distribution.

is distributed according to $F_{p,n} \in \Delta[0,\infty]$ where realization ∞ denotes a lack of report altogether.[4] I restrict attention to symmetric equilibria, and thus omit the firm's index from in much of the analysis below.

I place some restrictions on F. First, I assume that for all (p,n), $F_{p,n}$ must be piecewise twice differentiable and right-differentiable everywhere on $[0,\infty)$. This grants analytical convenience and ensures that equilibrium objects are well-defined.[5] Second, I impose a selection criterion (SC): a firm immediately reports a story it knows is true. This is stated as Definition 1.

Definition 1. F satisfies (SC) if

$$F_{1,n}(0) = 1 \text{ for all } n \in \{1, ..., N\}.$$

This criterion rules out unintuitive equilibria with periods of silence supported by pessimistic off-path beliefs, i.e., beliefs that reports made during these gaps have little or no credibility. Furthermore, this assumption implies that fixing any starting belief p, all firms who have not yet reported and the consumer hold the same *common belief* about the state after t time has passed. This common belief is denoted by $p(t)$.

Defining strategies in this way, i.e. with a separate distribution for each (p,n), is convenient but introduces redudancy. Namely, for any (p,n) and $t>0$, $F_{p,n}$ and $F_{p(t),n}$ "overlap": both distributions specify the firm's reporting behavior at $(p(t+s),n)$ for any $s \geq 0$. Thus, I impose that the $F_{p,n}$ must be mutually consistent.[6] That is, at any (p,n) on-path, $F_{p,n}$ and $F_{p(t),n}$ must satisfy

$$F_{p(t),n}(s) = \frac{F_{p,n}(t+s) - F_{p,n}(t_-)}{1 - F_{p,n}(t_-)} \text{ for all } s \geq 0 \text{ whenever } F_{p,n}(t) < 1, \tag{2}$$

where $F_{p,n}(t_-) \equiv \lim_{\tau \uparrow t} F_{p,n}(\tau)$. This formula is a result of Bayes Rule.

[4] By defining strategies in this way, firms can react instantly to a competitor's report. For instnace, if $F_{p,2}(t) = 1$ for all t and p, then if some firm makes the first report at t, all other firms will also report at t.

[5] Note that F satisfies the above restrictions if and only if there exist two functions on p, q_n and b_n, where for all (p,n) and $t \geq 0$,

$$F_{p,n}(t) = \sum_{s \leq t | q_n(p(s))>0} q_n(p(s)) + \int_0^t b_n(p(s))ds$$

such that b_n are piecewise differentiable and $q_n(p) = 0$ at all but a countable number of p. Namely, q_n denotes the *point mass* of reports, while b_n denotes the right *arrival rate* of reports.

[6] This condition is analogous to the closed-loop property specified in Fudenberg and Tirole (1985). I adopt the term *consistency condition* from Laraki, Solan, and Vieille (2005), who define this condition for a general class of continuous-time games of timing.

Before proceeding, I define two terms to describe reporting: faking and truth telling. A report is fake if it is made despite the firm lacking independent confirmation, i.e., a signal $s^i \neq \emptyset$. Meanwhile, a report that is made after the firm has confirmation is *truthful*. I also use these terms to describe reporting behavior: a firm is faking if it is sending a fake reports, while it is truth telling if its reports are exclusively truthful. Note that under the selection assumption (SC), strategies differ only in their distributions over fake reports.

I seek a symmetric perfect Bayesian equilibrium of this game. This is a Markov strategy F paired with beliefs α and p at each history such that F is sequentially rational and the beliefs are consistent with Bayes Rule. The consistency of α with Bayes Rule implies the following at all (p, n) on-path: [7]

$$\alpha_n(p) = \begin{cases} \frac{\lambda p}{\lambda p + b_n(p)} & \text{if } F_{p,n}(0) = 0 \\ 0 & \text{if } F_{p,n}(0) > 0 \end{cases} \tag{3}$$

where $b_n(p) \equiv F'_{p,n}(0+)$ is the right-derivative of $F_{p,n}$ at 0. That is $b_n(p)$ is the instantaneous arrival rate of fake reports.

This formula is intuitive. If $F_{p,n}(0) > 0$, there is a point mass of fake reports at (p, n). Meanwhile, because conclusive signals are distributed continuously over time, the instantaneous probability of a truthful report is zero. So, the consumer and all competing firms are certain that a report made at (p, n) was fake, and thus assign to it zero credibility. If there is not a point mass of fake reports at (p, n), credibility is assessed by comparing the arrival rates of truthful reports (λp) to that of fake reports ($b_n(p)$), assigning higher credibility to reports made when the arrival rate of fake reports is relatively low.

3. Equilibrium characterization

This section presents the equilibrium characterization. I first establish two properties that are instrumental to the analysis. Then, as a stepping stone to the full model characterization, I consider the monopoly case. This elucidates the forces at play even when competition is absent. In particular, I show that errors may occur even without competition, and that such errors are driven by a lack of commitment power by the firm. Finally, I characterize the equilibrium of the full model, establishing existence and uniqueness.

[7] The formula is derived by applying Bayes Rule to a discrete-time approximation of the beliefs that obtain under this game. This derivation is presented in Appendix B.

3.1. Properties of equilibrium

I begin by presenting two necessary conditions on a firm's equilibrium strategy that are crucial to the analysis that follows. First, I show that there cannot exist any jumps (i.e., point masses) in the distribution of fake reports. Second, whenever a firm is less-than-fully credible, it must satisfy certain indifference conditions. Notably, similar properties arise in other games with continuous strategy spaces, where they result from competition.[8] But as I will illustrate below, in this game they are instead driven by the endogenous nature of credibility and thus hold even in the absence of competition.

First, let us consider the "no jumps" property, which is formalized as Lemma 1:

Lemma 1. *In equilibrium, at any (p, n) on-path, $F_{p,n}$ is continuous everywhere whenever $p < 1$.*

This states that fake reports are distributed continuously over time whenever a firm is not certain that the story is true. I.e., there can never be a point mass in faking when $p < 1$. To see why, recall that a report made when there is a point mass of faking yields zero credibility. Meanwhile, faking while also not being certain than the story is true yields a strictly positive expected penalty $\beta(1 - p)$. Thus, a firm's value from faking at such a time is strictly negative. The firm could then profitably deviate by truth telling: this would preclude the firm from making an error, ensuring a weakly positive payoff.

Now let us state the "indifference" property. To do so, I must introduce two pieces of notation. Let δ_s for $s \in [0, \infty]$ denote the pure strategy that places full mass on faking after s time has passed. In particular, δ_0 denotes immediate faking, while δ_∞ denotes truthfulness (i.e., that the firm never fakes). Next, let $V_{p,n}$ denote the firm's value at (p, n) and function $V_{p,n}(\cdot)$ denote the value conditional on playing a certain strategy. That is, with this notation in hand, we are ready to state Lemma 2.

Lemma 2. *In equilibrium if $\alpha_n(p) < 1$ and (p, n) is on-path, then there exists an $\varepsilon > 0$ such that*

$$V_{p,n} = V_{p,n}(\delta_s) \textit{ for all } s \in [0, \varepsilon) \cup \infty.$$

Lemma 2 states that whenever $\alpha_n(p) < 1$, the firm must find a number of strategies optimal. First, it must be optimal to fake immediately (i.e., play δ_0). Second, it must be optimal to be truthful for some sufficiently short span of time dt and then fake (i.e., play δ_{dt}). Third, it must be optimal to never fake (i.e., play δ_∞). I will now provide some insight into the proof of this lemma, which is presented formally in the appendix. Let us begin by

[8] In particular, similar properties have been established in war of attrition games (Hendricks, Weiss, and Wilson (1988)) and all-pay auctions (Baye, Kovenock, and De Vries (1996)).

considering why δ_{dt} must be optimal for $dt \in [0, \varepsilon]$. It follows from our regularity conditions on the firm's strategy that $\alpha_n(p(s))$ must be right-continuous in s. This means that if $\alpha_n(p) < 1$, $\alpha_n(p(s)) < 1$ for all s sufficiently small. Furthermore, we recall that whenever $\alpha_n(p(s)) \in (0, 1)$, the firm is faking with a strictly positive hazard rate. This means that the firm mixes between faking with delay $[0, \varepsilon]$, implying that all such pure strategies are optimal. Next, let us consider why never faking (playing δ_∞) must be optimal, too. For some intuition, suppose by contradiction this were not the case. Then in equilibrium, a firm who has not received a conclusive signal must fake with probability 1. Because there are no point masses in the distribution of fake reports (Lemma 1), the firm must sustain a sufficiently high hazard rate of faking, even as the common belief p gets arbitrarily small. But because the hazard rate of truthful reports tends to zero as p shrinks, this means that credibility must deteriorate over time to a level that would make faking sub-optimal for the firm. It thus follows that with positive probability, the firm will remain truthful for the entirety of the game. While not immediately obvious, these indifference conditions are essential to the equilibrium characterization. In particular, I will show that they imply a boundary value problem that fully characterizes faking in equilibrium.

3.2. The monopoly characterization and the role of commitment

I now characterize the equilibrium under a monopoly, i.e., assuming $N = 1$. As there is only one firm, I drop the n index from all functions and parameters.

Claim 1. *Under a monopoly, for all p on-path*

$$\alpha(p) = \min\{\beta/k, 1\}.$$

From Claim 1, we can make a number of observations about the monopoly equilibrium. First, credibility must be constant over time. Second, a monopolist is not always fully credible. In particular, credibility is weakly increasing in k/β, and less-than-perfect whenever k/β is sufficiently small. That is, errors can occur when the ex-post penalty of error is relatively low. While not immediate, the constant nature of credibility implies dynamics in the the firm's strategy: a firm who fakes with positive probability becomes strictly more truthful over the course of time. In the remainder of this subsection, I provide intuition for properties and argue that a monopolist's errors are driven by their inability to commit.

Let us first consider why credibility is constant. Recall from Lemma 2 that whenever $\alpha_n(p) < 1$, the firm must be indifferent between faking immediately and after some short wait dt. By the martingale property of firm's belief p, both of these strategies yield the same expected penalty from error $\beta(1-p)$. In non-technical terms: a firm who waits before faking might learn the story is true and thus incur no penalty, but in the absence of a conclusive signal its belief that the story is true will fall. This rise and fall balance out so that the ex-ante cost of error does not change. Given this, for both strategies to be optimal, they must also yield the same expected prize $k\alpha$. Thus, credibility must not change. It is noteworthy that this reasoning implicitly assumes waiting is costless. Indeed, this is true under a monopoly. Not only is waiting intrinsically costless (i.e., future payoffs are not discounted), a monopolist does not incur the strategic cost to waiting that preemption might entail. As I show in Section 4, this strategic cost of waiting is precisely what gives rises to dynamics in credibility when there are multiple firms in the market.

While a monopolist's credibility is constant, its strategy exhibits dynamics: a firm who fakes will become gradually more truthful over time. Specifically, the hazard rate of faking (b) strictly decreases and tends to zero whenever credibility is less-than-perfect. This follows from the formula for credibility, (3). To see why, note that the more time passes without a report, the consumer becomes increasingly skeptical of the story. This declining belief is an artifact of the firm's one-sided Poisson learning process: the absence of a report means that the firm has not received a conclusive signal, an event that is consistent with $\theta = 0$.[9] In light of this, to ensure credibility remains constant, the hazard rate of fake reports must decline as well and tend to zero.

Now, I explain why truth telling cannot be sustained when β is relatively small, and that credibility increases in β. Suppose by contradiction that the firm is truthful. This implies that the firm is fully credible in equilibrium, and thus that that the market share ($k\alpha$) exceeds the penalty of error (β). So it is strictly optimal to report, even if the story is false. That is, the firm can profitably deviate by faking. We conclude that in any equilibrium, the firm must fake with positive probability. To pin down what credibility looks like, we recall from Lemma 2 that a firm who fakes must be indifferent between faking immediately and remaining truthful. Indeed, there is a unique value of credibility that ensures this indifference: β/k. There is some intuition behind this: the bigger β/k is, the more costly errors are compared to market share for any given α. Resultedly, the more costly faking is to truth-telling. So as β increases, α must correspondingly increase to maintain indifference between the two.

In this model, I assume that firms cannot commit to a reporting strategy. That is, I take the stance that firms can always deviate from their strategy, for instance by faking, after consumers have assessed credibility (ex-interim). Intuitively, this ex-interim temptation to fake can cause a deterioration of credibility in equilibrium. I show that taking away this

[9] In the monopoly case, $p(t) = \frac{pe^{-\lambda t}}{pe^{-\lambda t} + (1-p)}$.

temptation, i.e. allowing the monopolist to commit, errors would never take place.

To understand why, suppose that the the firm could commit to its strategy at the start of the game, and cannot deviate once credibility had been assessed.[10] One can immediately see that given the ability to commit, the firm would always choose truth telling over its non-commitment equilibrium strategy, even when $\beta < k$. By committing to truth telling, the firm is guaranteed a payoff of k if $\theta = 1$, and 0 if $\theta = 0$. Meanwhile, under the no-commitment equilibrium, the firm will earn strictly less when $\theta = 1$, due to its strictly lower credibility and earn 0 when $\theta = 0$: though the firm may fake as its payoff from faking exactly offsets the penalty of error. Intuitively, committing to truth-telling is better for the firm because the enhanced credibility the firm enjoys when the story is true exceeds any payoff it might enjoy from reporting a false story (which is nothing in equilibrium). One can strengthen this result, showing that truth telling maximizes the firm's expected payoff, and is thus the unique commitment solution under a monopoly (this result is presented formally in Appendix D). In other words, a monopolist who can commit will never err. This illustrates an important point about faking: it not only deteriorates the quality of information consumers receive, it also hurts firms. Despite this, faking happens because firms are unable to credibly promise truthfulness to consumers unless errors are costly enough to discipline the firm.

3.3. Full model characterization

Now, I characterize the equilibrium of the full model (i.e., under an arbitrary N), establishing both existence and uniqueness. To this end, I show that any equilibrium is the solution to a recursive set of boundary value problems. Specifically, whenever the firm is not truthful, credibility must satisfy an ODE and a boundary condition.

Let us begin by deriving the conditions under which the firm is truthful. This both serves as a stepping stone to a full characterization and illustrates how competition can deteriorate credibility and exacerbate faking. This result is stated as Proposition 1.

Proposition 1. *In equilibrium, at any (p, n) on-path, $\alpha_n(p) = 1$ if and only if:*

1. $k_n \leq \beta$
2. $p \leq p_n^* \equiv \min\{\frac{k_n - \beta}{\frac{k_n}{N-n+1} - \beta}, 1\}$.

Proposition 1 provides two conditions, on the model parameters and the common belief, that are necessary and sufficient for truth telling. Recall that the first condition alone,

[10] While we discuss the commitment solution informally here, a formal treatment is presented in Appendix D.

$k_n \leq \beta$, was sufficient for truth telling under a monopoly (Claim 1). However, under competition, a second condition is required: the common belief must lie below some threshold p_n^*. That is, firms must be sufficiently skeptical about the story's truth.

The need for this additional condition alone illustrates an important point: truth telling is harder to sustain under competition. Under a monopoly, the only cost to truth telling is the absence of market share when the story is false. But under competition, truth telling also entails a risk of being preempted. Namely, a truthful firm risks its opponents reporting first, either because they have learned the story is true or because they are faking. Taking for granted that being preempted is costly, it is easy to see why truth telling is harder to sustain.

But in this model, we cannot take for granted that being preempted is costly. It is, however, true that preemption is costly conditional on being truthful in equilibrium. This is most obvious in a *winner takes all* setting, where $k_n = 0$ for all $n > 1$. That is, all firms with the exception of the first to report earn zero market share. In this case, the costliness of being preempted is an artifact of the parameters, as a preempted firm can earn at best zero payoff. Generally the decreasing nature of k_n alone does not imply that preemption is costly: improved credibility for succeeding firms could endogenously counteract the decay in k_n, making preemption costless or even valuable. Indeed, I will show in the next section that under certain parameters, precisely such a phenomenon occurs in equilibrium. But conditional on a firm being truthful in equilibrium, this cannot happen: truthfulness implies full credibility, leaving no room for a succeeding firm to improve on it.

Let us now consider why under competition, truthtelling is only possible when firms are sufficiently pessimistic about the story. This can be explained by the fact that faking and truth telling each pose a different kind of risk to the firm: while truth telling entails the risk of being preempted, faking entails the risk of making an error and incurring penalty β. Both of these depend on the belief p about the state: higher p implies a lower probability of error but also a higher probability of being preempted. The former is obvious, and the latter is due to the fact that preemption is more likely when the story is true. Namely, conditional on the story being true, an opponent reports not just because it is faking, but also because it has received confirmation. Thus, a firm with a higher p will believe its risk of being preempted is higher, too. Because a lower risk of error and higher risk of preemption make faking relatively more profitable, truth telling is harder to sustain when p is high.

While Proposition 1 pins down the conditions under which the firm is truthful, it remains to characterize the firm's behavior when truth telling does not hold. To this end, I obtain a key result: when the firm fakes, credibility must satisfy an ODE and limit condition.

Proposition 2. *In equilibrium, at all (p,n) on-path where $k_n \geq \beta$ or $p > p_n^* \equiv \frac{k_n-\beta}{k_n/n-\beta}$, the following ODE must be satisfied:*

$$\alpha_n'(p) = -\frac{1}{k_n(1-p)\alpha_n(p)}\frac{N-n}{N-n+1}[k_n\alpha_n(p) - V_{\tilde{p},n+1} - \beta(1-\alpha_n(p))(1-p)], \qquad \text{(ODE)}$$

where $\tilde{p} \equiv \alpha_n(p) + (1-\alpha_n(p))p$.

In addition, $\lim_{p\to 0+}\alpha_n(p) = \beta/k_n$ must hold if $k_n > \beta$ and $\lim_{p\to p_n^*+}\alpha_n(p) = 1$ if $k_n \leq \beta$.

The proof for Proposition 2 follows from the indifference condition established in Lemma 2. Namely, when credibility lies below one, the firm must be indifferent between faking immediately and after some short wait dt. Formally, there exists an $\varepsilon > 0$ such that $V_{p,n}(\delta_0) = V_{p,n}(\delta_{dt})$ for all $dt \in (0,\varepsilon]$. Taking a Taylor approximation of the firm's value from waiting, $V_{p,n}(\delta_{dt})$, yields the following:

$$V_{p,n}(\delta_{dt}) - V_{p,n}(\delta_0) = [\frac{dp}{dt}(k_n\alpha_n'(p)) - \frac{\lambda p(N-n)}{\alpha_n(p)}(V_{\tilde{p},n} - V_{\tilde{p},n+1})]dt + o(dt^2), \qquad (4)$$

where $\tilde{p}$ denotes the common belief in the immediate aftermath of being preempted at (p,n). Taking the limit of this approximation as $dt \to 0$ is precisely what yields (ODE). Equation (4) is intuitive. It states that waiting to fake, rather than faking immediately, has two consequences for the firm's payoff. The first is that α_n, and thus the payoff from reporting, may change. This change in credibility is approximated by $\frac{dp}{dt}(k_n\alpha_n'(p))dt$. The second consequence is that by waiting, the firm risks being preempted: with probability $\frac{\lambda p(N-n)}{\alpha_n(p)}dt$ it is preempted, in which case its expected payoff changes by $V_{\tilde{p},n} - V_{\tilde{p},n+1}$. I interpret this decrease in value as the firm's *cost of preemption*.

To understand this interpretation, let us examine the probability and cost of preemption more closely. As one might expect, the probability of being preempted is increasing in the number of rival firms ($N-n$) and the expected rate at which these rivals are able to confirm the story (λp). It is also decreasing in equilibrium credibility: less credible firms are more likely to fake, and thus more likely to preempt. Meanwhile, the cost of preemption is the difference between two values, $V_{\tilde{p},n+1}$ and $V_{\tilde{p},n}$. Specifically, $V_{\tilde{p},n+1}$ denotes the firm's continuation value in the event that it is preempted at (p,n). This value is taken at $(\tilde{p},n+1)$ because preemption affects both the firms order and the common belief. Namely, while the common belief was p prior to the rival firm's report, it increases to $\tilde{p} \equiv \alpha_n(p) + (1-\alpha_n(p))p$ in the immediate aftermath of the report. To understand this expression for $\tilde{p}$, note that a rival firm's report means one of two things: either the report was triggered by a conclusive signal, in which case the new belief should be 1, or it was fake, in which case the new report offers no new information and the belief remains p. Since faking is unobservable, the new common belief $\tilde{p}$ is an average of these two conditional beliefs, where the weight given to the report being informed is its credibility. Meanwhile, $V_{\tilde{p},n}$ denotes the continuation value conditional on not being preempted. Notably, this value is not assessed at belief prior to preemption p, but rather the posterior $\tilde{p}$. In this sense, $V_{\tilde{p},n+1} - V_{\tilde{p},n}$ denotes firm's *regret* from not having reported sooner once it has been preempted.

In addition to (ODE), Proposition 2 establishes one of two limit conditions. Which limit condition holds depends on the model parameters, and like (ODE), result from the firm's indifference condition. First consider the case where $k_n \leq \beta$. Recall from Proposition 1 that in this case, $\alpha_n(p) = 1$ whenever $p \leq p_n^*$. We must then have that $\alpha_n(p)$ limits to 1 as the belief approaches p_n^*. If it did not, then as the belief approached p_n^*, the firm could profitably deviate by not faking immediately, and rather waiting an infinitesimal period until p_n^* is reached to do so. In the case where $k_n > \beta$, the firm never truth tells in equilibrium, and thus the indifference condition must always be satisifed. As the common belief p approaches zero, a firm who fakes does so being increasingly certain that its report is erroneous, and will incur penalty β. Thus, the firm's payoff from faking limits to the following:

$$\lim_{p \to 0+} V_{p,n}(\delta_0) = k_n \lim_{p \to 0+} \alpha_n(p) - \beta.$$

Separately, Lemma 2 tells us that the firm must find truth telling (δ_∞) optimal as well. As $p \to 0+$, the value of truth telling tends to zero, as it becomes increasingly likely that the firm never reports. The limit condition in this case, $\lim_{p \to 0+} \alpha_n(p) = \beta / k_n$, is precisely what is needed to ensure indifference between faking and truth telling.

To take stock, Proposition 1 and Proposition 2 provide two necessary conditions on equilibrium credibility. They pin down the region in which truth telling occurs (Proposition 1), and show that otherwise, credibility must satisfy a recursive boundary value problem (Proposition 2). One can show that these two conditions are sufficient for an equilibrium as well, provided that the firms strategy is consistent with this credibility function.[11] To prove this result, one must show that if credibility statisfies these conditions, the firm cannot profitably deviate from the strategy that is consistent with this credibility. On the region where credibilty is perfect, a deviation would consist of faking. Proposition 1 establishes that such a strategy cannot be played in equilibrium, that is, the firm could profitably deviate by truth telling even when their opponents are faking (i.e., the risk of being preempted is higher) and credibility is less-than-perfect (i.e., the benefit of reporting

[11] This result is presented in the Appendix as Lemma 5.

is lower). Such a strategy thus could not be more profitable than truth telling when the firm's opponents are not faking, and credibility is perfect. On the region where $\alpha_n(p) < 1$, the firm's strategy involves mixing between faking and remaining truthful. This too must be optimal, because both (ODE) and the boundary conditions guarantee it. In particular, as we have argued above, a credibility function satisfies these conditions if and only if it implies indifference between faking and truth telling.

Thus, the equilibrium is fully characterized by the solution to a recursive boundary value problem. While I do not have a closed-form solution to this problem, I use the Picard-Lindelof theorem to establish existence and uniqueness. This result is stated as Theorem 1.

Theorem 1. *There is a unique equilibrium, where uniqueness applies at (p, n) on-path.*

4. Dynamics and herding

With the above characterization in hand, I now study the dynamics of reporting. I show that credibility gradually improves over time whenever preemption is costly, with discrete changes in reporting behavior triggered by the report of a rival firm. Under certain conditions, in particular when observational learning is sufficiently strong, firms herd on their opponents decisions to report as well as the timing of these reports. This is due to a *copycat effect*, wherein one report causes an immediate surge in faking by others.

The nature of these dynamics will hinge on whether the last firm fakes in equilibrium. Thus, I will discuss two separate cases in turn: the first in which the last firm is truthful ($k_N \leq \beta$) and the second in which the last firm fakes with positive probability ($k_N > \beta$). I will show that firms face a preemptive motive when the last firm is truthful, but this motive endogenously disappears otherwise.

To facilitate the analysis, denote a *subgame* by a pair (p, n), where p denotes the starting common belief and n the order of the next firm to report. I begin by showing that fixing a subgame, i.e. assuming that no new reports are made, credibility strictly improves over time whenever $\beta > k_N$:

Proposition 3. *If $\beta > k_N$, $\frac{d}{dt}\alpha_n(p(t)) > 0$ and $\frac{d}{dt}b_n(p(t)) < 0$ whenever $\alpha_n(p(t)) < 1$, for all (p, n) on-path.*

The broad implication of this result is that while credibility is constant under a monopoly, competition can give rise to dynamics. To understand why, it can be helpful to observe the following: as long as $\alpha_n(p(t))$ has not reached its upper bound of 1, it must strictly increase precisely when there is a positive cost to preemption. Formally, this follows from (ODE). It is especially clear when we write (ODE) in the following form:

$$\frac{d}{dt}\alpha_n(p(t)) = \frac{\lambda p(N-n)}{\alpha_n(p(t))k_n}[V_{\tilde{p},n} - V_{\tilde{p},n+1}]. \tag{5}$$

We can see that $\alpha_n(p(t))$ is strictly increasing if and only if the cost of preemption, $V_{\tilde{p},n} - V_{\tilde{p},n+1}$, is strictly positive. There is intuition behind this result. Whenever the firm is less-than-fully credible, it must be indifferent between faking immediately and waiting some period of time before doing so. However, if credibility remained constant reporting immediately would be strictly better —it would allow the firm to avoid being preempted while suffering no harm to its credibility. To restore indifference, the firm must somehow be compensated for waiting. This can only be achieved by means of increasing credibility: while by waiting the firm risks being preempted, it will enjoy higher credibility otherwise. That is, credibility must increase to mitigate the haste-inducing effects of preemptive risk.

We have argued that credibility must increase when preemption is costly. However, as discussed above, this is not necessarily true even when there are multiple firms in the market. But it is indeed true that preemption is costly when $\beta > k_N$, i.e. when β is high enough to ensure the last firm to report is truthful. The proof requires a backwards induction argument, but its core reasoning is most easily illustrated in a duopoly setting ($N = 2$) where $\beta \in (k_2, k_1)$. In this case, a firm fakes with a positive hazard rate as long as nobody has reported yet, but switches to truth telling as soon as their opponent makes a report. Proposition 3 asserts that the credibility of the first report α_1 must strictly increase over time. To see why, suppose instead that α_1 is constant, as in the monopoly case.[12] Since $k_1\alpha_1(p(t))$ must limit to β (Proposition 3), it follows that $k_1\alpha_1(p(t)) = \beta$ for all t. That is, the market share from reporting first is always β, no matter when the report is made. This implies a failure of the firm's indifference condition: the market share from reporting first is so high that faking is strictly optimal. Specifically, if the story is false both faking and truth telling yield 0 payoff, but if the story is true the firm is ensured a payoff of β by faking but by truth telling risks being preempted and only earning k. To restore indifference, the market share of the first firm must instead be strictly less than β and approach it from below. This restores indifference because it increases the value of truth-telling in two ways: (1) the lower market share from reporting first lowers the cost of being preempted and (2) as argued above, increasing α provides an additional incentive to wait.

[12] Note that the arugment we make here is purely illustrative; it does not rule out the possibility that $\alpha_1(p(t))$ is increasing in t, nor that the function is only locally non-increasing. For a formal treatment, please see the proof.

Note that Proposition 3 also states that the hazard rate of faking, $b_n(p(t))$, is decreasing in t, an immediate corollary of the increasing nature of credibility. While this same result obtains in the monopoly case, the strictly increasing nature of credibility implies that $b_n(p(t))$ decays more quickly than under the constant credibility of the monopoly equilibrium. I.e., the firm's preemptive motive also gives rise to more extreme dynamics in faking.

So far, we have restricted attention to the case where $k_N < \beta$. In fact, one can show that if this does not hold, preemption becomes costless in equilibrium and the dynamics in α_n disappear. I formalize this result as Proposition 4.

Proposition 4. *If $k_N > \beta$, $\alpha_n(p) = \beta/k_n$ for all (p, n).*

This result states that when $k_N \geq \beta$, credibility is constant at a level where the market share $\alpha_n(p)k_n$ is not affected by the firm's order. That is, firms enjoy higher credibility from reporting after their opponents, which mitigates the decline in k_n in such a way that makes preemption costless.

To understand the reasoning for this claim, it is again helpful to consider the duopoly case, but this time assuming that $\beta < k_2 < k_1$. It follows from the monopoly characterization that the market share for the second reporter, $k_2\alpha_2(p)$, equals β no matter when that report is made. Now let us consider the first reporter. Again, the market share of the first reporter must limit to β. But in this case, it cannot limit to β from below. If it did, a firm could profitably deviate by not being truthful: being preempted would *benefit* the firm, as it would yield a higher market share β. Instead, the market share of the first firm, $k_1\alpha_1(p)$, must always equal β: this ensures that the firm incurs no loss in value by being preempted, and so its indifference condition is preserved.

Before proceeding, let us take stock of these results. Proposition 3 asserts that under certain conditions, news reports that are made with greater delay for research are more trustworthy to consumers. I.e., all else equal, consumers will have greater trust in a firm's journalistic standards when a report is not made quickly. In this sense, this model provides justifation for consumer distrust of hasty reporting that originates from the firm's preemptive motive. Meanwhile, Proposition 4 establishes a notable feature of the equilibrium: competition alone does not imply preemptive concerns. Even though reporting first yields more market share, all else equal, payoffs may endogenously adjust in such a way that makes preemption costless. Furthermore, because the firm's continuation value is determined by backwards induction, the existence of a preemptive motive hinges on the incentives of the last firm to report.

Proposition 3 and Proposition 4 describe the dynamics of reporting conditional on no new reports being made. For a more complete picture of equilibrium dynamics, it is helpful

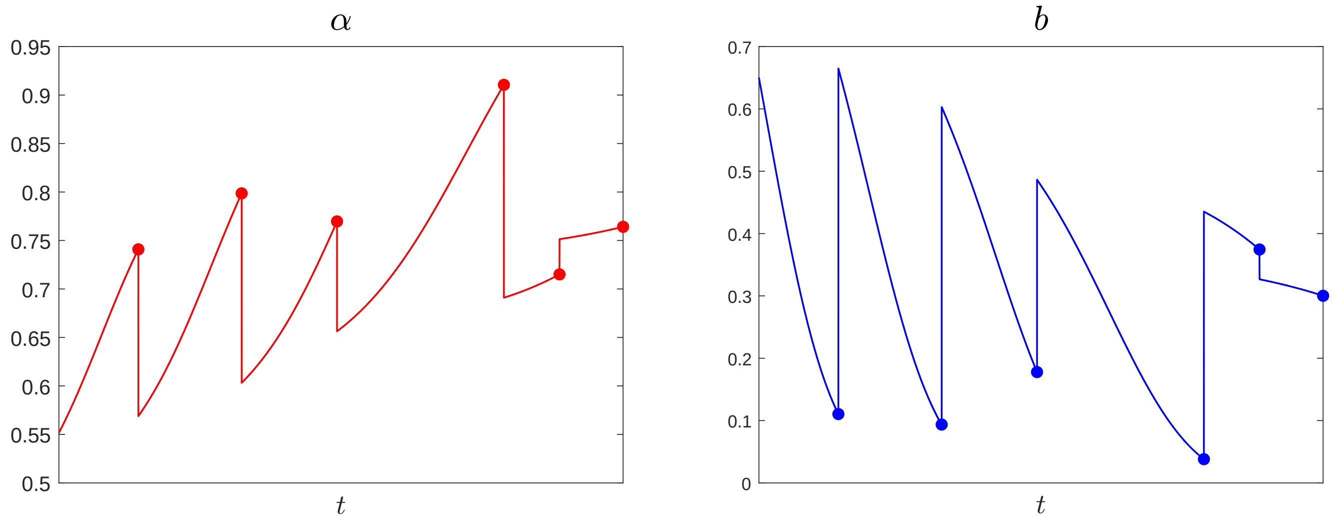


Figure 1: Simulation of crediblity (α) and the hazard rate of faking (b), over the course of a game when $k_N < \beta$. Discrete jumps signify that a firm has made a report. Upwards jumps in b illustrate the copycat effect.

to plot simulations of credibility and faking over the course of time. Figure 2a does this for the case when $k_N < \beta$. As per Proposition 3, credibility is continuously increasing and faking continuously decreasing as long as no new reports are made. However, we can see that a new report triggers discrete jumps in credibility and faking. Notably, as illustrated by these graphs, these jumps need not be monotonic.

Dynamics are qualitatively different when $k_N \geq \beta$. This is illustrated by Figure 2a, which plots a simulation in this case of the parameters. As per Proposition 4, credibility is flat, with new reports triggering exclusively upwards jumps. But despite this, faking exhibit a discrete upwards jump in the aftermath of a new report.

Both these simulations illustrate a *copycat effect*, in which one firm's report causes a surge in the hazard rate of faking for others. This is significant because it implies herding on the decision to report a story, both when the story is true and when it is false. Furthermore, because this surge in faking is immediate, the copycat effect is also consistent with herding on the *timing* of news reports, a phenomenon which is documented in the empirical literature. The remainder of this section is dedicated to understanding the forces responsible for the copycat effect, and the conditions in which it is most prevalent.

Let us identify the forces responsible for the copycat effect. To this end, note that the discrete change in faking following a report at (p, n) is given by

$$b_{n+1}(\tilde{p}) - b_n(p).$$

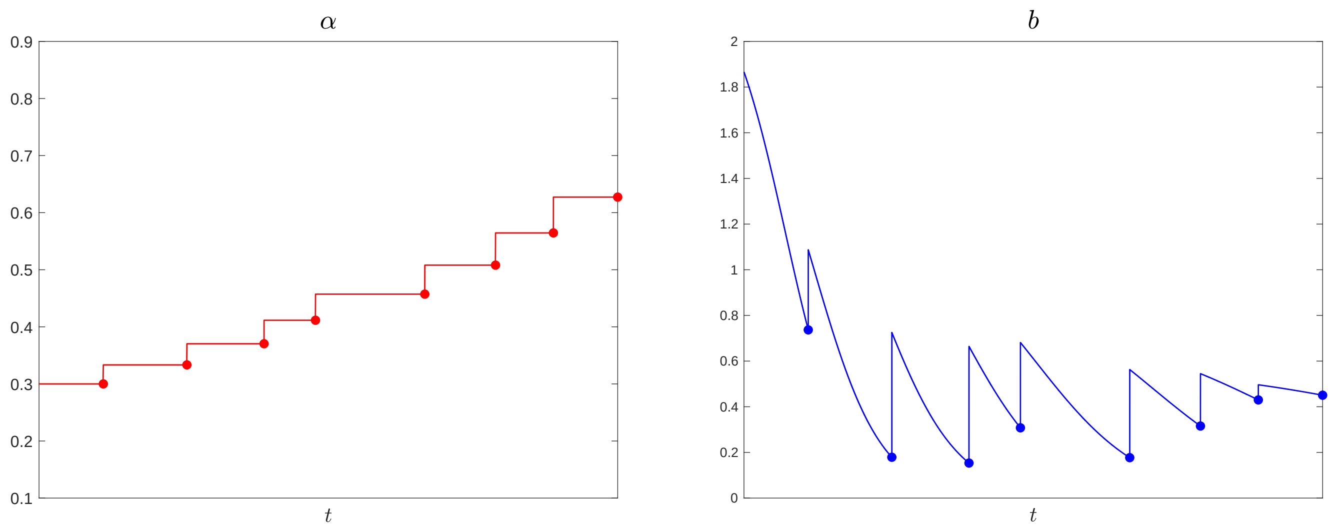


Figure 2: Simulation of crediblity (α) and the hazard rate of faking (b), over the course of a game when $k_N > \beta$. Upwards jumps in b illustrate the copycat effect.

This expression shows that a new report affects two changes to the state. First, it changes the firm's order if they choose to report: rather than being the n^{th} firm, they will be the $n+1^{th}$ firm. Second, the common belief increases from p to $\tilde{p}$: firms become more confident that the story is true after observing the opponent's report. The following decomposition isolates the respective impacts of these two changes:

$$b_{n+1}(\tilde{p}) - b_n(p) = \underbrace{[b_{n+1}(p) - b_n(p)]}_{\text{change in order}} + \underbrace{[b_{n+1}(\tilde{p}) - b_{n+1}(p)]}_{\text{change in belief}}$$

The change in order has an ambiguous effect on faking, i.e., $b_{n+1}(p) - b_n(p)$ may be positive or negative. Intuitively, this is because a report by one firm can cause either an increase or decrease in the firm's preemptive motive. To illustrate this, it is helpful to study two simple examples. First, consider a three firm setting $(N = 3)$ where $k_1 > k_2 = k_3$ and $\beta \in (k_3, k_1)$. In this case, firms face a cost to preemption as long as nobody has yet reported, but this cost disappears once at least one firm has reported since the firm's order will no longer impact its market share ($k_2 = k_3$). So, a change in the firm's order reduces its incentive to fake. Next, consider the same example but now assume that $k_1 = k_2 > k_3$. In this case, all else equal, the first and second reporter enjoy the same market share ($k_1 = k_2$). So, firms face no cost to preemption as long as nobody has reported yet. Instead, this cost materializes as soon as the first report has been made. So in this case, a change in order increases the incentive to fake.

Unlike the change in order, observational learning causes an unambiguous increase in faking. That is, $b_{n+1}(\tilde{p}) - b_{n+1}(p)$ is strictly positive whenever firms fake. Formally, this is an immediate result of Proposition 3 and Proposition 4, which establish that whenever a firm fakes, $b_n(p(t))$ is decreasing in t. Because the common belief $p(t)$ decays with time, this means $b_n(p)$ is increasing increasing in p. That is, a jump in the common belief implies an increase in faking. There is intuition for this as well: all else equal, a firm that is more optimistic has a greater incentive to fake. As discussed earlier, this is due to the fact that a higher p corresponds to a lower risk of error and higher risk of being preempted, both of which make faking relatively more valuable.

The copycat effect has important implications for the behavior of firms. It implies that in the aftermath of an opponent report, a firm is more likely to report not because they have also uncovered the truth, but because they are faking. If a report had not been made, however, the firms' hazard rate of reporting would continue to decrease. That is, the copycat effect is consistent with herding on firms' decision to report. Furthermore, because firm faking increases immediately and then starts its gradual decline, a new report is most likely in the immediate aftermath of an opponent report. That is, this behavior is consistent with herding on the timing of reports. While one might conceive that such herding might be the result of firms receiving correlated signals, the copycat effect provides us with a strategic motive for such behavior.

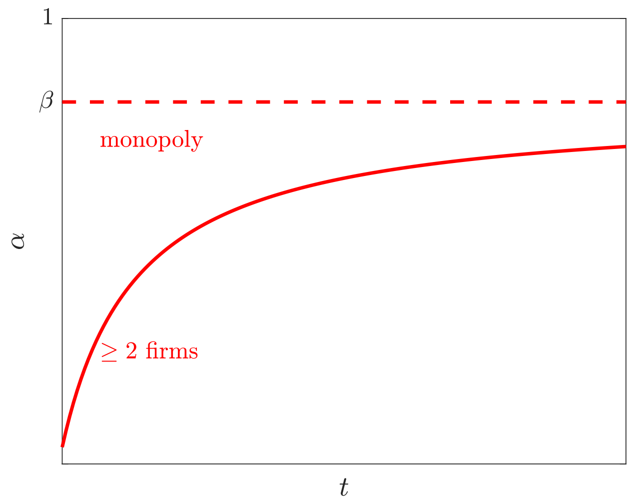
5. Media mergers

In this section, I consider the implications of this model for media mergers. I find that while a merger may improve credibility early on in the news cycle, this comes at the expense of lower-credibility reporting later in the news cycle.

Formally, I compare the equilibrium under $N \geq 2$ firms (*pre-merger*) to that under a monopoly (*post-merger*), holding fixed the market's total ability to learn and total maximal market share. I.e., I assume that each firm has ability λ pre-merger and the monopolist has ability $N\lambda$ post-merger. This normalization is motivated by the fact that merging news firms ostensibly combine their news rooms, and thus their capacities for research. Further, I assume the n^{th} firm enjoy a maximal market shares k_n pre-merger, while the monopolist enjoys a maximal market share $k^m \equiv \sum_{n=1}^{N} k_n$ post-merger. This is consistent with the notion that merging news firms combine their consumer bases.[13] Finally, I assume that the cost of error (β) and the prior about the story (p_0) is the same pre- and post-merger.

[13] For a formal justification for why this implies that the maximal market share would be $\sum_{n=1}^{N} k_n$, please see the payoff function microfoundation.

Notably, defining a merger in this way is restrictive: it rules out the possibility that only a subset of firms merge. While ideally, one would allow for such partial consolidation of the market, I restrict attention to complete consolidation purely for tractibility and believe it nonetheless provides insight into the effects of mergers more generally.



The merger affects reporting by both changing the credibility of the first report and eliminating the possibility of succeeding reports. Let us begin by considering the former, i.e. the impact of the merger on the first news report. It follows from the above characterization that if the prior about the story being true (p_0) is sufficiently high, reporting may be more credible early on in the news cycle. However, later reports will always be less credible post-merger whenever not in a winner-takes-all setting. This second point is formalized as Corollary 1.

Corollary 1. *If $\beta \in (k_1, k_N)$, there exists a $\overline{p} > 0$ such that $\alpha_1^m(p) < \alpha_1(p)$ for all $p < \overline{p}$, where α and α^m denote the pre- and post-merger equilibrium credibility, respectively.*

These findings are illustrated by **??**, which plots a simulation of credibility in the market conditional on the time of the first report, assuming that $\beta \in (k_1, k_N)$. Here, reports are more credible post-merger for small t (i.e., with little time for research), but are less credible once the common belief falls below a threshold. To understand why this is the case, one must recall that the merger consists of two changes to the market. First, any preemptive motive firms may have faced before is eliminated. This elimination of the preemptive motive reduces the incentive to fake, and is why post-merger credibility may be higher early when the common belief is still relatively high. But in addition to this, the post-merger firm also enjoys a greater maximal market share. All else equal, this makes faking more profitable, and thus causes a deterioration in credibility. While both changes occur simluateneously, the credibility-improving effects of eliminating the preemptive motive disappears as the common belief falls. such credibility-improving effects are limited to instances where firms face a cost to preemption pre-merger: because there is no cost to preemption when $\beta \leq k_N$, post-merger credibility will be lower no matter when the report is made.

This result has implications for the impact of media mergers on news quality. While previous research has argued that media mergers can exacerbate bias (Anderson and McLaren (2012)) and ideological persuasion (Balan, DeGraba, and Wickelgren (2003)), the above analysis suggests that the effects on factual errors are more nuanced. While mergers can improve credibility by eliminating firms' preemptive motive, such improvement can only occur early on in a news cycle and if firms are sufficiently optimistic about the story to begin with. Otherwise, the effects of market consolidation will cause credibility to suffer. Beyond affecting the quality of the first report, the merger comes at the cost of reports by succeeding firms. Although such succeeding reports may suffer in credibility due to the copycat effect, they nonetheless serve as an additional signal about the story.

6. Comparative Statics

In this section, I consider how the equilibrium changes with the parameters of the model. This will shed light on how different features of the news market can either exacerbate or curb erroneous reporting. These findings are stated as Proposition 5.

Proposition 5. *In any equilibrium, for any n, $\alpha_n(p(t))$ is*

(a) weakly increasing in β, and strictly so whenever $\alpha_n(p(t)) < 1$.

(b) weakly increasing in λ, and strictly so for $t > 0$ whenever $\alpha_n(p(t)) < 1$ and $k_N < \beta$.

(c) weakly decreasing in N, and strictly so whenever $\alpha_n(p(t)) < 1$, when $t \in [0, \bar{t}]$ for some $\bar{t} > 0$.

Part (a) states that no matter when a firm reports, it will be more credible under high β. This result is intuitive: a higher ex-post cost of error means firms are less likely to fake, and thus more credible. This result is a consequence of the firm's equilbrium incentives: a higher β makes faking more costly. This will either induce the firm to resort to truth telling instead, or require that it is compensated for this coster faking with greater credibility.

Now, let us consider the comparative static on λ. It states that credibility is higher whenever firms have a greater ability to learn. Let us now understand what is driving this result.

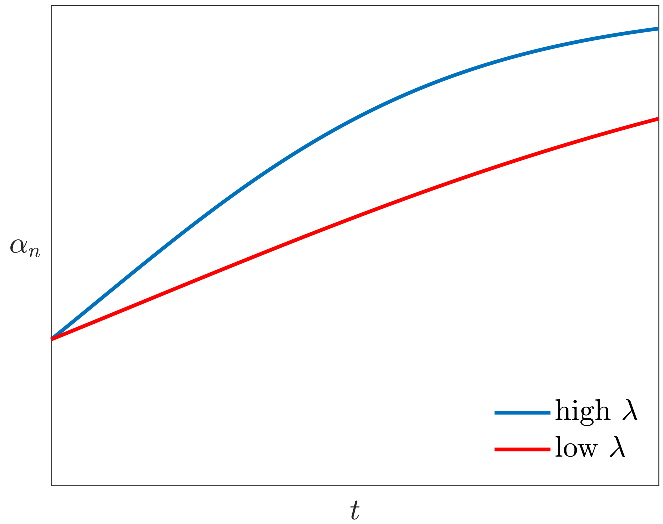


Figure 3: A simulation of $\alpha_n(p(t))$ when $\lambda = 1$ (blue line) and $\lambda = 0.5$ (red line). For the remaining parameter values, the following specifications were made: $\beta = 0.5$, $p_0 = 0.7$, $N = 8$, $k_n = 0.7^{(N-n)}$.

We first note that at any belief p the firm may hold, a change in λ will have *no effect* on $\alpha_n(p)$ in equilibrium. This is due to the fact that λ does not enter the boundary value problem which dictates the firm's credibility, and thus changes in λ have no effect $\alpha_n(p)$. However, changes in λ will have an effect on the time path of the common belief $p(t)$. Under a higher λ, firms learn about the state more quickly, and thus $p(t)$, the belief that $\theta = 1$ conditional on no reports, will decay faster. That is, firms will be more pessimistic about the story's validity at any time $t > 0$ when λ is higher. This greater pessimism about the story translates to a higher expected cost of erring, which makes faking more costly. As was true of the comparative static on β, this increased cost of faking must be counterbalanced by a higher credibility $\alpha_1(p(t))$ at every time $t > 0$ to ensure indifference holds. This comparative static is illustrated by Figure 3, which shows simulations of the firm's credibility function under both high and low values of λ.

Let us finally consider the comparative static on the total number of firms, N. While it pertains to the level of competition, this exercise is notably distinct from our analysis in the previous section. Therein, we studied the overall impact of competition on equilibrium outcomes. This was done by comparing the case where competition is present ($N > 1$) to the monopoly case ($N = 1$) while holding constant the total learning ability of the market, $N\lambda$. With this comparative static, we are instead considering the *marginal* impact of an additional firm entering the market. In particular, we do not hold fixed the total learning ability of the market. Rather, I assume that this additional firm adds to the total learning

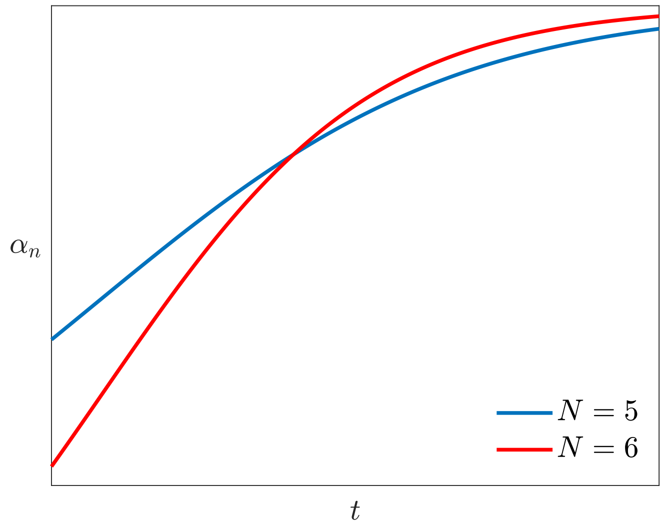


Figure 4: A simulation of $\alpha_n(p(t))$ when $N = 5$ (blue line) and $N = 6$ (red line). For the remaining parameter values, the following specifications were made: $\beta = 0.5$, $p_0 = 0.7$, $\lambda = 1$, $k_n = 0.7^{(N-n)}$.

ability of the market. In doing so, one can study the effect of *proliferation* in the news industry.

Proposition 5 states that adding a firm to the market will guarantee a deterioration in credibility, but only for a limited amount of time. In fact, the addition of a firm may result in an improvement in credibility during later periods. This phenomenon is captured by Figure 4. This figure plots simulations of α_n under $N = 5$ and $N = 6$, respectively, holding all other parameters fixed. While the addition of a firm lowers credibility in early periods, it improves credibility in later periods.

To understand this result, note that an additional firm will affect two separate changes to the market. First, each firm faces greater competition, and thus a greater risk of being preempted. This change is precisely what was captured in our earlier exercise regarding the effects of competition. As illustrated by **??**, this change will cause a deterioration in credibility. However, an additional firm also increases the market's total ability to learn. This change is captured by our comparative static on λ, which shows that an increase in learning ability will cause an improvement in credibility. Thus, the effect of an additional firm can be understood as the combination of two countervailing forces: higher competition and a higher ability to learn within the market.

To understand why the credibility-diminishing effect of higher competition must dominate in early periods, we must compare the relative magnitudes of these the two countervailing forces. Figure 3 illustrates that while credibility is pointwise higher at every $t > 0$ under high λ, this difference is negligible in early periods. This is due to the fact that firms learn gradually over time, and thus it takes time for differences in learning ability to substantially impact firms' beliefs. Meanwhile, as illustrated by **??**, an increase in competition will have a non-negligible impact on credibility even when $t = 0$. For this reason, the impact of higher competition must dominate in early periods, resulting in a net reduction in credibility. However, as time passes and the effect of faster learning grows, a reversal may take place, i.e., there may be a net improvement in credibility. Such a scenario is precisely what is depicted by Figure 4.

7. Extension: Heterogeneous Ability

I consider an extension in which firms are heterogeneous in their abilities to learn. Doing so will shed light on how a firm's credibility correlates with it's ability in equilibrium.

This extended model is identical to the model above except for three changes. First, rather than assuming that each firm is endowed with the same ability λ, I assume that each firm i is endowed with an firm-specific ability λ^i. As with all other parameters, I assume that these firm-specific abilities are common knowledge. Second, to simplify our analysis for this exercise, I will restrict attention to a winner-takes-all setting: i.e., I assume that $k_n = 0$ for all $n > 1$. Finally, I relax our assumption that the equilibrium is symmetric. Thus, different firms (and in particular, firms with different abilities) may play different strategies in equilibrium and thus a firm's credibility is firm-specific. Accordingly, I let α^i denote the credibility of firm i.

I obtain an intuitive result: a firm's ability correlates positively with its credibility in equilibrium. This is stated as Proposition 6.

Proposition 6. *For all (i, j) such that $\lambda^i < \lambda^j$, $\alpha_1^i(p(t)) \leq \alpha_1^j(p(t))$. Furthermore, this inequality is strict whenever $\alpha_1^i(p(t)) < 1$.*

Proposition 6 states that regardless of when a report is made, a firm with higher ability will be more credible.[14] Furthermore, a high ability firm will be strictly more credible than a low ability firm whenever firms are not fully truthful.

Let us now consider why this correlation arises. First, note that high ability firms are able to confirm a story more quickly and thus, all else equal, pose a greater preemptive

[14] This claim restricts attention to the first firm to report, because by the winner-takes-all assumption, all following senders will never fake, i.e., $\alpha_n^i(p) = 1$ whenever $n > 1$.

threat in equilibrium. This in turn implies that in comparison to a high-ability firm, a low-ability firm faces a greater preemptive threat. Thus, the low-ability firm finds immediate faking more advantageous. In light of this, the firms' credibilities must adjust in such a way to preserve their respective indifference conditions. This is achieved endogenously by means of a lower credibility for the low-ability firm, which ensures that it has less to gain from faking immediately.

8. Conclusion

In this paper, I presented a dynamic model of breaking news to understand the nature of reporting errors. I sought to explain how strategic forces that could induce firms to err. In this setting, errors were driven by two qualities of the breaking news environment: a firm's lack of commitment power as well as competition. I find that competition induces firms to err through two separate channels: preemptive motives and observational learning. While preemptive motives can give rise to errors by encouraging firms to report hastily, observational learning can cause an existing error to propagate through the market.

The second key objective was to understand the dynamics of reporting errors. In equilibrium, these dynamics take two forms. First, firms become gradually more truthful over time as long as no new reports are made. Furthermore, a firm's credibility gradually increases whenever preemptive motives are at play. Importantly, this improvement in credibility incentivizes firms to take their time, and thus counteracts the haste-inducing effects of preemption. Dynamics also take the form of discrete changes in the firm's behavior and credibility which are triggered by a rival report. In particular, I document a copycat effect, where a report by one firm can induce a surge in faking by other firms in the market.

While I consider breaking news specifically, this model provides broader insight into how preemptive concerns can affect the quality of information provided by experts. To understand how preemption impacts information provision more broadly is a topic that warrants further investigation.

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Appendix A Microfoundation for market share

In the main text, I assumed that the the firm's market share from reporting a story is $k_n\alpha$. Here I provide a microfoundation for this.

Let $\mathcal{N} \equiv \{1, ..., N\}$ denote the set of news firms. Suppose there is a mass $K > 0$ of consumers, who are indexed by x. Each consumer x *subscribes* to some subset S_x of the firms. I.e., for all x, $S_x \subseteq \mathcal{N}$. Let S_x denote consumer x's *subscription set*. Fixing any $S \subseteq \{1, ..., N\}$, let $m(S)$ denote the mass of consumers x such that $S_x = S$, where $\sum_{S\in 2^{\mathcal{N}}} m(S) = K$. Assume that the mass of consumers with a given subscription set does not depend on the identity of the firms within that set, but only on the number of firms in the set. Formally, suppose that there exists $\gamma_0, \gamma_1, ..., \gamma_N \geq 0$ such that

$$m(S) = \gamma_n \text{ if and only if } |S| = n, \text{ where } \sum_{n=1}^{N} \gamma_n \binom{N}{n} = K.$$

Define i's *market share* to be the mass of consumers who read the story. We assume that a consumer reads a story if she both *considers* the story, and finds it optimal to read it. To formalize this, let $\hat{S} \subseteq \mathcal{N}$ denote the set of firms who reported before i. A consumer x will *consider* a story if and only if:

1. The firm is in the consumer's subscription set, i.e., $i \in S_x$.
2. The consumer has not previously considered the story. I.e., $j \notin S_x$ for all $j \in \hat{S}$.

The mass of consumers who consider reading firm i's story is then given by

$$\sum_{j=1}^{N-n} \binom{N-n}{j} \gamma_{j+1} \equiv k_n,$$

where n is the order of i's report. Next, suppose consumer x faces a cost c_x of reading a story. Suppose that c_x is i.i.d. across x, that for any x, c_x is uniformly distributed on $[0, 1]$, and that c_x is independent of x's consideration set. Then x's payoff from reading a story is $\mathbb{I}[\theta = 1] - c_x$. That is, the consumer will incur a cost c_x from reading the story, and a benefit of 1 only if the story is true. Meanwhile, the consumer's payoff from not reading a story is $\mathbb{I}[\theta = 0]$. Namely, the consumer enjoys a payoff of 1 from refusing to story that is untrue. Assuming consumers maximize expected utility, x will read the story if and only if

$$\alpha + (1-\alpha)p - c_x \geq (1-\alpha)(1-p) \Leftrightarrow c_i \leq \alpha$$

where α is the credibility of i's story. Thus i's market share is $k_n \alpha$.

Appendix B Equilibrium credibility

Here, I justify equation (3) by showing that it is the limit of Bayes-consistent beliefs under a discrete approximation of the game presented in Section 2. To this end, for any $\varepsilon > 0$, let the *ε-approximation* of the game be identical to the game presented in section (2), except with the following modification: any report made by a firm on $[0, \varepsilon]$ is observed by all other players (including the consumer) at ε. That is, rather than observing t_i, the players observe $\tilde{t}_i$, where

$$\tilde{t}_i \equiv max\{t_i, \varepsilon\}$$

At any (p, n) that is on-path, let $\alpha_n^\varepsilon(p)$ denote the firm's credibility, i.e., the consumer's belief that $s_i \leq \varepsilon$ given that $\tilde{t}_i = \varepsilon$, under the ε-approximation of the game. Let α_n denote the right-limit of the α_n^ε. Then:

$$\alpha_n(p) \equiv \lim_{\varepsilon \to 0+} \alpha_n^\varepsilon(p)$$

I now establish that $\alpha_n(p)$ is given by (3) at any (p, n) on-path.

Claim 2. *For any (p, n) on-path,*

$$\alpha_n(p) = \begin{cases} \frac{\lambda p}{\lambda p + b_n(p)} & \text{if } F_{p,n}(0) = 0 \\ 0 & \text{if } F_{p,n}(0) > 0 \end{cases}$$

Proof. For any $\varepsilon > 0$, it follows from Bayes Rule that

$$\alpha_n^\varepsilon(p) = \frac{p(1 - e^{-\lambda\varepsilon})}{p(1 - e^{-\lambda\varepsilon}) + F_{p,n}(\varepsilon)e^{-\lambda\varepsilon}}.$$

If $F_{p,n}(0) = 0$, it follows from L'Hôpital's Rule that:

$$\lim_{\varepsilon \to 0+} \alpha_n^\varepsilon(p) = \frac{\lambda p}{\lambda p + b_n(p)}$$

If $F_{p,n}(0) > 0$, it follows from the right-continuity of $F_{p,n}$ that

$$\lim_{\varepsilon \to 0+} \alpha_n^\varepsilon(p) = \frac{0}{0 + \lim_{\varepsilon \to 0+} F_{p,n}(\varepsilon)} = 0.$$

□

Appendix C Equilibrium characterization

C.1 The firm's problem

Here, I formally state the firm's problem. I begin by defining a useful object, the *first report distribution*. Fix a report history H and strategy profile F, and let p denote the common belief and n the order of the next firm to report. Index the firms who have not yet reported by i. Let the first report distribution $\Psi^i(s)$ denote the probability that player i reported at or before s time passes and was not preceded by any of the remaining firms in doing so.[15] This is given by:

$$\Psi^i(s) = p \int_0^s e^{-\lambda r(N-n)} \prod_{j \neq i} (1 - F_{p,n}^j(r)) d(e^{-\lambda r}(F_{p,n}^i(r) - 1)) + (1-p) \int_0^s \prod_{j \neq i} (1 - F_{p,n}^j(r)) dF_{p,n}^i(r).$$

Define the firm's problem recursively as follows. Fix a history H and corresponding (p, n). Fix a firm i who has not yet reported, and assume all $j \neq i$ play the same strategy F^{-i}.[16] Let $V_{p,n}(F^i)$ denote i's payoff from playing strategy F^i. It follows that

$$V_{p,n}(F^i) = \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s) + (N-n) \int_0^\infty V_{p^{-i}(s),n+1} d\Psi^{-i}(s), \tag{6}$$

where Ψ^{-i} denotes the first-report distribution for any $j \neq i$. Note that first integral of this expression is firm i's expected payoff from reporting when it is the first of the remaining firms to do so, and the second integral is the firm's expected payoff conditional on being preempted. The firm's problem at (p, n) is then given by the following:

$$\max_{F^i \in \mathcal{F}} V_{p,n}(F^i),$$

where $\mathcal{F}$ denotes the set of permissible distributions, i.e., those that are piecewise continuously differenetiable, right-differentiable, and that satisfy the selection criterion (SC). Further, let $V_{p,n} \equiv \sup_{F^i \in \mathcal{F}} V_{p,n}(F^i)$.

[15] While Ψ is a function of F, p, and n, I omit this dependences for brevity.

[16] We can make this assumption because we restrict attention to symmetric equilibria.

C.2 Proofs

Proof of Lemma 1. Let us begin by showing that at all (p, n) on-path such that $p < 1$, $F_{p,n}$ is continuous at 0. To this end, suppose by contradiction that $F_{p,n}$ is discontinuous at 0. By the right-continuity of $F_{p,n}$, this implies that $F_{p,n}(0) > 0$. Because (p, n) is on path, by (3), $\alpha_n(p) = 0$. Furthermore, it follows by (**??**) that $p^i(0) = p$. Recalling that we are restricting attention to symmetric equilibria, let Ψ denote the first-report distribution at (p, n) under the equilibrium strategy profile $F_{p,n}$. Because $F_{p,n}(0) > 0$, $\Psi^j(0) > 0$ for all j who have not yet reported.

Now define the following deviation $\hat{F}_{p,n}$. This strategy is identical to $F_{p,n}$, except that all the mass that $F_{p,n}$ places on 0 is shifted to ∞:

$$\hat{F}_{p,n}(s) = \begin{cases} F_{p,n}(s) - F_{p,n}(0) & \text{if } s < \infty \\ 1 & \text{if } s = \infty \end{cases}$$

Now, fix some i who has not yet reported. Let $\hat{\Psi}$ denote the first-report distribution at (p, n) under the strategy profile where i plays $\hat{F}_{p,n}$ and all $j \neq i$ play $F_{p,n}$. By definition, for all $s \geq 0$,

$$\hat{\Psi}^i(s) = \Psi^i(s) - \Psi^i(0).$$

Then,

$$\begin{aligned} \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1 - p^i(s))] d\hat{\Psi}^i(s) &= \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s) + \beta(1 - p^i(0))\Psi^i(0) \\ &> \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s). \end{aligned}$$

Again by definition, for all $s \geq 0$,

$$\hat{\Psi}^{-i}(s) = \Psi^{-i}(s) + X(s),$$

where

$$\begin{aligned} X(s) \equiv \Psi^i(0)[p \int_0^s (1 - F_{p,n})^{n-2}(1 - \hat{F}_{p,n}(r)) e^{-\lambda r} d(e^{-\lambda r}(F_{p,n}(r) - 1)) \\ +(1 - p) \int_0^s (1 - F_{p,n}(r))^{n-2}(1 - \hat{F}_{p,n}(r)) dF_{p,n}(r)] \end{aligned}$$

Then, we have

$$\int_0^\infty V_{p^{-i}(s),n+1} d\hat{\Psi}^{-i}(s) - \int_0^\infty V_{p^{-i}(s),n+1} d\Psi^{-i}(s) = \int_0^\infty V_{p^{-i}(s),n+1} dX(s) \geq 0.$$

where the final inequality follows from the fact that $X(s)$ is increasing in s and $V_{p^{-i}(s),n+1} \geq V_{p^{-i}(s),n+1}(\delta_\infty) \geq 0$.

Combining the above two inequalities we have

$$V_{p,n}(\hat{F}_{p,n}) = \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\hat{\Psi}^i(s) + (N-n) \int_0^\infty V_{p^{-i}(s),n+1} d\hat{\Psi}^{-i}(s)$$
$$> \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s) + (N-n) \int_0^\infty V_{p^{-i}(s),n+1} d\Psi^{-i}(s) = V_{p,n}(F_{p,n})$$

Thus, i can profitably deviate at (p,n). Contradiction.

We will now show that for all (p,n) on-path such that $p < 1$, $F_{p,n}$ must be continuous at all t. Suppose by contradiction that it is not. Let t denote the time at which a discontinuity occurs. Because $F_{p,n}$ is increasing and right-differentiable by assumption, this must be a jump discontinuity, i.e.,

$$\lim_{r \to t-} F_{p,n}(r) < F_{p,n}(t)$$

By (2),

$$F_{p(t),n}(0) = \frac{F_{p,n}(t) - \lim_{r\uparrow t} F_{p,n}(r)}{1 - \lim_{r \uparrow t} F_{p,n}(r)} > 0.$$

But then, this implies that $F_{p(t),n}$ is discontinuous at 0, contradicting the above.

Part (b) of the statement follows directly from (3). □

Lemma 3. *For any (p,n) on-path,*

- $\alpha_n(p) \geq \overline{\alpha}_n(p) \equiv \min\{\beta(1-p)/k_n, 1\}$
- $F'_{p,n}(0+) \leq \overline{f} \equiv \lambda p(\frac{1}{\overline{\alpha}_n(p)} - 1)$

Proof of Lemma 3. We begin by showing the first point above. The second point follows by definition of $\alpha_n(p)$.

First, suppose by contradiction that there exists a (p,n) on-path such that

$$\alpha_n(p) < \min\{\beta(1-p)/k_n, 1\}$$

Recalling that $p(s)$ is given by (**??**), we begin by claiming that for all s sufficiently small, $(p(s), n)$ is on-path. Suppose not by contradiction. Since (p,n) is on-path by assumption, this implies that $F_{p,n}(s) = 1$, which contradicts Lemma 1. It thus follows from (3), combined with the piecewise twice differentiability and right-differentiability of $F_{p,n}$, that $\alpha_n(p(s))$ is continuous in some right-neighborhood of $s = 0$. I.e., there exists an $\varepsilon > 0$ such that for all $s \in [0, \varepsilon]$,

$$k_n \alpha_n(p(s)) < \beta(1-p).$$

Next, I claim that $F_{p,n}(\varepsilon) > 0$. Suppose this is not true by contradiction. Then, it follows that $F_{p,n}(s) = 0$ for all $s \in [0, \varepsilon]$, implying by definition of α that $\alpha_n(p) = 1$, contradicting our assumption that $\alpha_n(p) < 1$.

Now, define the following deviation $\tilde{F}_{p,n}$, which shifts the mass $F_{p,n}$ places on $[0, \varepsilon]$ to ∞:

$$\tilde{F}_{p,n}(s) = \begin{cases} 0 & \text{if } s \in [0, \varepsilon] \\ F_{p,n}(s) - F_{p,n}(\varepsilon) & \text{if } s \in (\varepsilon, \infty) \\ 1 & \text{if } s = \infty \end{cases}$$

The admissibility (i.e., right-continuity and piecewise twice-differentiability) of $\tilde{F}_{p,n}$ follows from the admissibility of $F_{p,n}$. We now wish to show that $\tilde{F}_{p,n}$ is a profitable deviation at (p, n). Let Ψ denote the first-report distribution under the strategy profile where all players play $F_{p,n}$, and let $\tilde{\Psi}$ denote the first-report distribution under the strategy profile where i plays $\tilde{F}_{p,n}$ and all $j \neq i$ play $F_{p,n}$.

By definition of Ψ,

$$\tilde{\Psi}^i(s) = \Psi^i(s) - X(s)$$

where

$$X(s) = \begin{cases} p \int_0^s e^{-\lambda r(N-n)}(1 - F_{p,n}(r))^{N-n} d(e^{-\lambda r}(F_{p,n}(r) - 1)) + (1-p)\int_0^s (1 - F_{p,n}(r))^{N-n} dF_{p,n}(r) & \text{if } s \in [0, \varepsilon] \\ X(\varepsilon) & \text{if } s > \varepsilon \end{cases}$$

Now, note that $X(s)$ is weakly increasing in s. Note further that because $F_{p,n}(\varepsilon) \in (0, 1]$, it follows that $F_{p,n}(s)$ strictly increases on $[0, \varepsilon]$. Thus, $X(s)$ is strictly increasing at some $s \in [0, \varepsilon]$. Now, by the above definition:

$$\begin{aligned} \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\tilde{\Psi}^i(s) - \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s) \\ = \int_0^\varepsilon [k_n \alpha_n(p(s)) - \beta(1 - p(s))] dX(s) > 0 \end{aligned}$$

where the strict inequality follows from the fact that $X(s)$ is strictly increasing on $[0, \varepsilon]$ and the above-established fact that $k_n\alpha_n(p(s)) < \beta(1-p(s))$ for all $s \in [0,\varepsilon]$.

Next, let us consider $\tilde{\Psi}^{-i}(s)$. It again follows from the definition of Ψ that

$$\tilde{\Psi}^{-i}(s) = \Psi^{-i}(s) - Y(s)$$

where

$$Y(s) = p\int_0^s [e^{-\lambda r}(1-F_{p,n}(r))]^{n-2} F(\min\{r,\varepsilon\}) d(e^{-\lambda r}(F_{p,n}(r)-1)) + \\ (1-p)\int_0^s (1-F_{p,n}(r))^{n-2} F_{p,n}(\min\{r,\varepsilon\}) dF_{p,n}(r)$$

Thus,

$$\int_0^\infty V_{p^{-i}(s),n+1} d\tilde{\Psi}^{-i}(s) - \int_0^\infty V_{p^{-i}(s),n+1} d\Psi^{-i}(s) = \int_0^\infty V_{p^{-i}(s),n+1} dY(s) \geq 0$$

where the final inequality follows for from the fact that $Y(s)$ is increasing in s and $V_{p^{-i}(s),n+1} \geq 0$. Combining the previous two inequalities, we obtain that

$$V_{p,n}(\tilde{F}_{p,n}) > V_{p,n}(F_{p,n})$$

and thus i can profitably deviate at (p,n). Contradiction. $\square$

Proof of Lemma 2. Assume that $\alpha_n(p) < 1$. Note that by the right twice-differentiability of $F_{p,n}$, and by (3), that $\alpha_n(p(s))$ is right-continuous in s. Thus, there exists an $\varepsilon > 0$ and $d > 0$ such that

$$\alpha_n(p(s)) < 1-d \text{ for all } s \in [0,\varepsilon].$$

I claim that for all $s \in [0,\varepsilon)$, $V_{p,n} = V_{p,n}(\delta_s)$. Suppose to the contrary that for some $s \in [0,\varepsilon)$,

$$V_{p,n}(\delta_s) < V_{p,n}$$

Now, I claim that $V_{p,n}(\delta_s)$ is right-continuous in s. To see why this is the case, note that by definition,

$$V_{p,n}(\delta_s) = \int_0^s k_n\alpha_n(p(r)) d\Psi^i(r) + (N-n)\int_0^s V_{p^i(r),n} d\Psi^{-i}(r) + \\ (1-\sum_j \Psi^j(s))[k_n\alpha_n(p(s)) - \beta(1-p(s))]$$

Where $\Psi^j(s)$ is the first-report distribution that arises when i plays δ_∞ and all $j \neq i$ play

$F_{p,n}$. The right-continuity with respect to s then follows from the absolute continuity of Ψ^j (which follows from Lemma 1), as well as the right-continuity of $\alpha_n(p(s))$ with respect to s, which follows from the right-continuity of $F_{p,n}(s)$ by assumption.

Given the right continuity of $V_{p,n}(\delta_s)$, there exists some $\varepsilon' \in (0, \varepsilon - s)$ and $x > 0$ such that

$$V_{p,n} - V_{p,n}(\delta_r) > x \text{ for all } r \in [s, s+\varepsilon']$$

Now I claim that there must exist some $s^* \in [0, \infty]$ such that $V_{p,n} = V_{p,n}(\delta_{s^*})$. Suppose by contradiction that $V_{p,n} > V_{p,n}(\delta_{s^*})$ for all $s^* \in [0, \infty]$. Letting $F_{p,n}$ denote the firm's equilibrium strategy, it follows that $V_{p,n} = V_{p,n}(F_{p,n})$. It follows from (6) that

$$V_{p,n}(F_{p,n}) = \int_0^\infty V_{p,n}(\delta_s) dF_{p,n}(s) + (1 - \lim_{s\to\infty} F_{p,n}) V_{p,n}(\delta_\infty) < V_{p,n}$$

where the strict inequality follows from the assumption that $V_{p,n} > V_{p,n}(\delta_{s^*})$ for all s^*. Contradiction.

Now, define the following deviation $\tilde{F}_{p,n}$ which shifts all the mass from $[s, s+\varepsilon']$ to s^*. Specifically, when $s^* < s$:

$$\tilde{F}_{p,n}(t) = \begin{cases} F_{p,n}(t) + F_{p,n}(s+\varepsilon) - F_{p,n}(s) & \text{if } t \in [s^*, s] \\ F_{p,n}(s+\varepsilon) & \text{if } t \in (s, s+\varepsilon'] \\ F_{p,n}(t) & \text{otherwise.} \end{cases}$$

Meanwhile, when $s^* > s + \varepsilon$:

$$\tilde{F}_{p,n}(t) = \begin{cases} F_{p,n}(s) & \text{if } t \in [s, s+\varepsilon] \\ F_{p,n}(t) - [F_{p,n}(s+\varepsilon') - F_{p,n}(s)] & \text{if } t \in (s+\varepsilon', s^*) \\ F_{p,n}(t) & \text{otherwise.} \end{cases}$$

Now, by definition:

$$V_{p,n}(\tilde{F}_{p,n}) = V_{p,n}(F_{p,n}) + \int_s^{s+\varepsilon'} [V_{p,n}(\delta_{s^*}) - V_{p,n}](\delta_r) dF_{p,n}(r) \geq V_{p,n}(F_{p,n}) + x\varepsilon' > V_{p,n}(F_{p,n})$$

Thus, $\tilde{F}_{p,n}$ is a profitable deviation. Contradiction.

It remains to show that $V_{p,n} = V_{p,n}(\delta_\infty)$. Suppose by contradiction that $V_{p,n} > V_{p,n}(\delta_\infty)$. It follows that $\lim_{t\to\infty} F_{p,n}(t) = 0$, because otherwise, the firm could profitably deviate by placing no mass on $t=\infty$. But this implies that for some $s \in (0,\infty]$,

$$\lim_{t \to s-} b_n(p(t)) = \infty \Rightarrow \lim_{t \to s-} \alpha_n(p(t)) = 0,$$

which contradicts Lemma 3. $\square$

Lemma 4. *$\alpha_n(p(s))$ is continuous in s for all (p,n) on path such that $s > 0$.*

Proof of Lemma 4. Fix a (p,n) on-path. I first claim that for all $s \geq 0$,

$$\alpha_n(p(s)) = \frac{\lambda p(s)}{\lambda p(s) + \frac{F'_{p,n}(s+)}{1-F_{p,n}(s)}} \tag{7}$$

To see why, note that it follows from Lemma 3 that $(p(s), n)$ is on-path for all $s \geq 0$. Thus, by Lemma 1, $F_{p(s),n}(0) = 0$, and by (3)

$$\alpha_n(p(s)) = \frac{\lambda p(s)}{\lambda p(s) + F'_{p(s),n}(0+)}.$$

Next, it follows from (2) that

$$F'_{p(s),n}(0+) = \frac{F'_{p,n}(s+)}{1-F_{p,n}(s)}.$$

Combining the previous two equations yields (7). It thus follows from the right-differentiability and piecewise twice-differentiability of $F_{p,n}$ that $\alpha_n(p(s))$ is right-continuous in s. It remains to show that it is left-continuous. Suppose by contradiction there exists an s such that $\alpha_n(p(s))$ is left-discontinuous. Then there exists some $d > 0$ such that for all $\varepsilon > 0$, there exists an $s_\varepsilon \in (s-\varepsilon, s)$ such that

$$|\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s))| > d.$$

First consider the case where for all $\varepsilon > 0$, there exists an $s_\varepsilon \in (s-\varepsilon, s)$ such that $\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s)) > d$. I begin by claiming that for all $\varepsilon > 0$,

$$V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}). \tag{8}$$

To this end, first note that there exists some $s^* \in (s,\infty]$ such that $V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_{s^*})$. To see why this must hold, suppose not, by contradiction. Then it must be that $F_{p(s_\varepsilon),n}$ places full mass on $[s_\varepsilon, s]$, and thus, either Lemma 1 or (3) would be violated. Thus, we have

$$V_{p(s_\varepsilon),n} = \int_0^{s-s_\varepsilon} k_n \alpha_n(p(r)) d\Psi^i(r) + (N-n) \int_0^{s-s_\varepsilon} V_{p^i(r),n+1} d\Psi^{-i}(r) +$$

$$(1 - \sum_j \Psi^j(s-s_\varepsilon)) V_{p(s),n}(\delta_{s^*-s}) = \int_0^{s-s_\varepsilon} k_n \alpha_n(p(r)) d\Psi^i(r) + (N-n) \int_0^{s-s_\varepsilon} V_{p^i(r),n+1} d\Psi^{-i}(r)$$

$$+ (1 - \sum_j \Psi^j(s-s_\varepsilon)) V_{p(s),n}(\delta_0) = V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon})$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p(s_\varepsilon),n}$. Note that the equality follows from the fact that $\alpha_n(p(s)) < 1$, and thus by Lemma 2, $V_{p(s),n} = V_{p(s),n}(\delta_0)$. However, note that for all $\varepsilon > 0$,

$$V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = \int_0^{s-s_\varepsilon} k_n \alpha_n(p(r)) d\Psi^i(r) + (N-n) \int_0^{s-s_\varepsilon} V_{p^i(r),n+1} d\Psi^{-i}(r)$$
$$+ (1 - \sum_j \Psi^j(s-s_\varepsilon))[k_n \alpha_n(p(s), n) - \beta(1-p(s))]$$

Because the Ψ^j are absolutely continuous,

$$\lim_{\varepsilon \to 0} V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = k_n \alpha_n(p(s), n) - \beta(1-p(s))$$

Then, by the assumption that $\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s)) < d$, for all $\varepsilon > 0$ sufficiently small $V_{p(s_\varepsilon),n}(\delta_0) = k_n \alpha_n(p(s_\varepsilon), n) - \beta(1-p(s_\varepsilon)) > V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon})$, contradicting (8).

Next, consider the case where for all $\varepsilon > 0$, $\alpha_n(p(s)) - \alpha_n(p(s_\varepsilon)) > d$. As noted above, $\lim_{\varepsilon \to 0} V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = V_{p(s),n}(\delta_0)$. Thus, for ε sufficiently small,

$$V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) > k_n \alpha_n(p(s_\varepsilon)) - \beta(1-p(s_\varepsilon)) = V_{p(s_\varepsilon),n}(\delta_0)$$

However, since $\alpha_n(p(s_\varepsilon)) < 1$ for all $\varepsilon > 0$, by Lemma 2, $V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_0)$. Contradiction. □

Proof of Proposition 1. I begin by showing that $\alpha_n(p) = 1$ whenever $k_n < \beta$ and $p \leq p_n^* \equiv \frac{k_n - \beta}{k_n/n - \beta}$. To this end, fix an n, and suppose that $k_n < \beta$. I first show that for all $q < \frac{\beta - k_n}{\beta}$, $\alpha_n(q) = 1$. Note that for all such q

$$V_{q,n}(\delta_0) = k_n \alpha_n(q) - \beta(1-q) \leq k_n - \beta(1-q) < k_n - \beta(1 - \frac{\beta - k_n}{\beta}) = 0.$$

Since $V_{q,n} \geq V_{q,n}(\delta_\infty) \geq 0$, it follows $V_{q,n} > V_{q,n}(\delta_0)$. Thus, by Lemma 2, $\alpha_n(q) = 1$. Now, let

$$q_n^* \equiv \sup\{p | \alpha_n(q) = 1 \text{ for all } q < p\}$$

It follows from the above that $q_n^* \geq \frac{\beta - k_n}{\beta} > 0$. Now suppose by contradiction that $q_n^* < p_n^*$. By Lemma 4, there exists an $\varepsilon > 0$ such that for all $p \in (q_n^*, q_n^* + \varepsilon)$, $\alpha_n(p) < 1$, and thus, by Lemma 2

$$V_{p,n} = V_{p,n}(\delta_0) = k_n \alpha_n(p) - \beta(1-p)$$

Thus, it follows from Lemma 4 that

$$\lim_{p \to q_n^* +} V_{p,n} = k_n - \beta(1 - q_n^*) \tag{9}$$

By definition of V, because by Lemma 1 $F_{p,n}$ is absolutely continuous, it follows that $V_{p,n}(\delta_\infty)$ is as well, and thus:

$$\lim_{p \to q_n^* +} V_{p,n}(\delta_\infty) = V_{q_n^*,n}(\delta_\infty) = \frac{k_n q_n^*}{n} \tag{10}$$

In order for δ_∞ to not serve as a profitable deviation for $p \in (q_n^*, q_n^* + \varepsilon)$, it must be that for all such p, $V_{p,n}(\delta_0) \geq V_{p,n}(\delta_\infty)$. Taking a limit we obtain that

$$\lim_{p \to q_n^* +} V_{p,n}(\delta_0) \geq \lim_{p \to q_n^* +} V_{p,n}(\delta_\infty)$$

Substituting (9) and (10) above, we obtain that $\frac{k_n q_n^*}{n} \leq k_n - \beta(1 - q_n^*)$. However, $k_n \leq \beta$ and $q_n^* < p$ implies that $\frac{k_n q^*}{n} > k_n - \beta(1 - q^*)$. Contradiction.

Next, we show that $\alpha_n(p) < 1$ whenever $\beta \leq k_n$ or $p > p_n^*$. To this end, assume $\beta \leq k_n$ or $p > p_n^*$. Assume by contradiction that $\alpha_n(p) = 1$. Also assume by induction that if $n < N$, then the statement holds for $n + 1$.

First, consider the case where $\alpha_n(q) = 1$ for all $q < p$. By (3), this implies that $F'(q, n) = 0$. Furthermore, by Lemma 1, this implies that $F_{p,n}(s) = 0$ for all $s > 0$, i.e., $F_{p,n} = \delta_\infty$. However,

$$V_{p,n}(\delta_0) = k_n - \beta(1-p) > \frac{k_n p}{n} = V_{p,n}(\delta_\infty),$$

where the above strict inequality follows from the above assumption that either $\beta \leq k_n$ or $p > p_n^*$. Contradiction.

Next, consider the case where $\alpha_n(q) < 1$ for some $q < p$. By Lemma 4, for all $\varepsilon > 0$ sufficiently small, there exists some $\overline{p} < p$ and $\overline{s} > 0$ such that $\alpha_n(\overline{p}) \in (1 - \varepsilon, 1)$ and $\alpha_n(q)$

is strictly increasing on $[\overline{p}(\overline{s}), \overline{p}]$. By Lemma 2, there exists some $\Delta \in (0, s)$ such that

$$V_{\overline{p},n}(\delta_\Delta) = V_{\overline{p},n}(\delta_0).$$

By definition,

$$V_{\overline{p},n}(\delta_\Delta) = \int_0^\Delta k_n\alpha_n(\overline{p}(s))d\Psi^i(s) + (N-n)\int_0^\Delta V_{\overline{p}^i(s),n+1}d\Psi^{-i}(s) + \\ (1 - \sum_j \Psi^j(\Delta))(k_n\alpha_n(\overline{p}(\Delta)) - \beta(1-\overline{p}(\Delta))$$

where Ψ is the first-report distribution associated with the strategy profile where i plays δ_Δ and all $j = i$ play $F_{p,n}$. Meanwhile,

$$\begin{aligned} V_{\overline{p},n}(\delta_0) &= k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}) \\ &= \int_0^\Delta k_n\alpha_n(\overline{p})d\Psi^i(s) + (N-n)\int_0^\Delta k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}^i(s))d\Psi^{-i}(s) \\ &+ (1 - \sum_j \Psi^j(\Delta))(k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}(\Delta)) \end{aligned}$$

Thus, in order to preserve the above equality, for some $r \in (0, \overline{s})$,

$$k_n\alpha_n(\overline{p}) - \beta(1-p^i(r)) < V_{\overline{p}^i(r),n+1}. \tag{11}$$

First, consider the case where $\alpha_{n+1}(\overline{p}^i(r))) < 1$. Then, for $\varepsilon > 0$ sufficiently small

$$V_{\overline{p}^i(r),n+1} = V_{\overline{p}^i(r),n+1}(\delta_0) = k_{n+1}\alpha_{n+1}(\overline{p}^i(r)) - \beta(1-\overline{p}^i(r)) < k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}^i(r))$$

where the first equality follows from Lemma 2. Thus, equation (11) is violated. Contradiction.

Next, consider the case where $\alpha_{n+1}(\overline{p}^i(r)) = 1$ and $\beta < k_n$. By the inductive assumption, it follows that $\alpha_{n+1}(q) = 1$ for all $q \leq \overline{p}^i(s)$. Thus, $F_{\overline{p}^i(s),n+1} = \delta_\infty$. So, we have that for ε sufficiently small:

$$\begin{aligned} V_{\overline{p}^i(r),n+1} &= V_{\overline{p}^i(r),n+1}(\delta_\infty) = \frac{k_{n+1}\overline{p}^i(r)}{N-n} \leq \overline{p}^i(r)k_n\alpha_n(\overline{p}) + (1-\overline{p}^i(s))k_n\alpha_n(\overline{p}) - \beta) \\ &= k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}^i(s)) \end{aligned}$$

Again, this is a contradiction of (11).

Finally, consider the case where $\alpha_{n+1}(\overline{p}^i(r)) = 1$ and $\beta \geq k_n$. Recall by Proposition 1 that $\alpha_n(q) = 1$ for all $q \geq p_n^*$. Thus, because $\alpha_n(\overline{p}) < 1$, it follows from (4) that $\alpha_n(\overline{p}(s))$ must be strictly increasing in s for some $s > r$. Formally, let

$$r' \equiv \inf\{s > r | \alpha_n(\overline{p}(s)) \text{ is strictly increasing}\}.$$

First, I claim that

$$k_n \alpha_n(\overline{p}(r')) - \beta(1 - \overline{p}^i(r)) < V_{\overline{p}^i(r),n+1} \tag{12}$$

By the inductive assumption, since $\alpha_{n+1}(\overline{p}^i) = 1$, it must be that $\alpha_{n+1}(q) = 1$ for all $q < \overline{p}^i(r)$. Because $\alpha_n(\overline{p}(s))$ is weakly decreasing for $s \in [r, r']$, it follows by definition of $\overline{p}^i(s)$ that $\overline{p}^i(s) < \overline{p}^i(r)$ for all $s \in [r, r']$. Thus, for all $s \in [r, r']$

$$V_{\overline{p}^i(s),n+1} = \frac{k_{n+1}\overline{p}^i(s)}{N - n}.$$

It follows from this that for all $s \geq r$,

$$\begin{aligned} k_n \alpha_n(\overline{p}(s)) - \beta(1 - \overline{p}^i(s)) &< V_{\overline{p}^i(s),n+1} \\ \Leftrightarrow k_n \alpha_n(\overline{p}(s)) - \beta(1 - \overline{p}^i(s)) &< \frac{k_{n+1}\overline{p}^i(s)}{N - n} \\ \Leftrightarrow \overline{p}^i(s) &< \frac{\beta - k_n \alpha_n(\overline{p}(s))}{\beta - k_{n+1}/(N - n)} \end{aligned}$$

Now, because $\alpha_n(\overline{p}(s))$ is strictly decreasing on $s \in [0, r]$,

$$k_n \alpha_n(\overline{p}(r)) - \beta(1 - p^i(r)) < k_n \alpha_n(\overline{p}) - \beta(1 - \overline{p}^i(r)) < V_{\overline{p}^i(r),n+1}$$

where the second inequality holds by the same reasoning presented in the explanation for (11). Thus we have

$$\overline{p}^i(r') < \overline{p}^i(r) < \frac{\beta - k_n \alpha_{n+1}(\overline{p}(r))}{\beta - k_{n+1}/(N - n)} < \frac{\beta - k_n \alpha_{n+1}(\overline{p}(r'))}{\beta - k_{n+1}/(N - n)}$$

which implies (12).

It follows from this that there exists an $r'' > r'$ such that for all $s \in [r', r'']$, $\alpha_n(\overline{p}(s))$ is weakly decreasing and $V_{\overline{p}^i(s),n+1} > k_n \alpha_n(\overline{p}(r')) - \beta(1 - p^i(s))$. I now claim that

$$V_{\overline{p}(r'),n}(\delta_0) < V_{\overline{p}(r'),n}(\delta_{r''-r'}).$$

To see why, note that by definition,

$$V_{\overline{p}(r'),n}(\delta_{r''-r'}) - V_{\overline{p}(r'),n}(\delta_0) = \int_{r'}^{r''} k_n[\alpha_n(p(s)) - \alpha_n(p(r'))]d\Psi^i(s) +$$
$$\int_{r'}^{r''} [V_{p^i(s),n+1} - (k_n\alpha_n(p(r')) - \beta(1-p^i(s)))]d\Psi^{-i}(s)$$
$$+ \sum_j (\Psi^j(r'') - \Psi^j(r'))k_n(\alpha_n(p(r'')) - k_n\alpha_n(p(r')))$$

Since $\alpha_n(p(s)) \geq \alpha_n(p(r'))$ and $V_{p^i(s),n+1} > k_n\alpha_n(p(r')) - \beta(1-p^i(s))$ $s \in [r', r'']$, it follows that $V_{\overline{p}(r'),n}(\delta_{r''-r'}) - V_{\overline{p}(r'),n}(\delta_0) > 0$. However, this contradicts Lemma 2. $\square$

Proof of Proposition 2. Proof by induction. Fix an n, and assume that $\alpha_m(p)$ satisfies the above for all $m > n$ such that (p, m) is on-path.

We begin by showing that (ODE) must hold whenever $\alpha_n(p) < 1$. To this end, assume that $\alpha_n(p) < 1$. Then, by Lemma 2, there exists an $\varepsilon > 0$ such that for all $\Delta \in (0, \varepsilon)$,

$$\frac{V_{p,n}(\delta_\Delta) - V_{p,n}(\delta_0)}{\Delta} = 0 \tag{13}$$

Recall that by definition of V, that

$$V_{p,n}(\delta_0) = k_n\alpha_n(p) - \beta(1-p).$$

Meanwhile

$$V_{p,n}(\delta_\Delta) = \int_0^\Delta k_n\alpha_n(p(s))\Psi^i(s)ds + (N-n)\int_0^\Delta V_{p^{-i}(s),n+1}\Psi^{-i}(s)ds +$$
$$(1 - \sum_j \lim_{s\to\Delta-} \Psi^j(s))[k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta))]$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F^j_{p,S}$. Specifically, for all $s > 0$,

$$\Psi^i(s) = p\lambda \int_0^s e^{-\lambda rn}(1 - F_{p,n}(r))^{N-n}dr$$

$$\Psi^{-i}(s) = p\int_0^s e^{-\lambda r(N-n)}(1-F_{p,n}(r))^{n-2}d(-e^{-\lambda r}(1-F_{p,n}(r))) + (1-p)\int_0^s (1-F_{p,n}(r))^{n-2}dF_{p,n}(r)$$

It follows from Lemma 1 that, for all j, Ψ^j is also absolutely continuous, I.e., there exists a function ψ_j such that:

$$\Psi^j(s) = \int_0^s \psi^j(r)dr.$$

Specifically, according to Lemma 1, one such ψ^i and ψ^{-i} are given by the following:

$$\psi^i(s) = p\lambda e^{-\lambda sn}(1 - F_{p,n}(s))^{N-n}$$

$$\psi^{-i}(s) = pe^{-\lambda sn}(\lambda + F'_{p,n}(s+) - \lambda F_{p,n}(s))(1 - F_{p,n}(s))^{n-2} + (1-p)(1 - F_{p,n}(s))F'_{p,n}(s+)$$

Substituting these expressions for both $V_{p,n}(\delta_0)$ and $V_{p,n}(\delta_\Delta)$ into (13) and rearranging, we obtain that for all $\Delta \in (0, \varepsilon)$,

$$K_1(\Delta) + K_2(\Delta) + K_3(\Delta) = 0 \tag{14}$$

where

$$K_1(\Delta) \equiv \frac{\int_0^\Delta k_n[(\alpha_n(p(s)) - \alpha_n(p)) + \beta(1-p)]\psi^i(s)ds}{\Delta}$$

$$K_2(\Delta) \equiv \frac{(N-n)\int_0^\Delta [V_{p^{-i}(s),n+1} - k_n\alpha_n(p) + \beta(1-p)]\psi^{-i}(s)ds}{\Delta}$$

$$K_3(\Delta) \equiv \frac{(1 - \sum_j \lim_{s\to\Delta-}\psi^j(\Delta))[k_n(\alpha_n(p(\Delta)) - \alpha_n(p)) + \beta(p(\Delta) - p)]}{\Delta}$$

Now, we consider $\lim_{\Delta\to 0+}$ of $K_1(\Delta)$, $K_2(\Delta)$, and $K_3(\Delta)$ separately.

For $K_1(\Delta)$, it follows from L'Hôpital's Rule, together with the continuity of $\alpha_n(p(\Delta))$ (i.e., Lemma 4) and $\psi^i(\Delta)$ in Δ that

$$\lim_{\Delta\to 0+} K_1(\Delta) = \lim_{\Delta\to 0+}[k_n(\alpha_n(p(\Delta)) - \alpha_n(p)) + \beta(1-p)]\psi^i(\Delta) = \beta(1-p)\psi^i(0) = \beta(1-p)p\lambda.$$

For $K_2(\Delta)$, it again follows from L'Hôpital's Rule, together with the right-continuity of $V_{p^{-i}(\Delta),n+1}$ in Δ that

$$\begin{aligned}\lim_{\Delta\to 0+} K_2(\Delta) &= (N-n)\lim_{\Delta\to 0+}[V_{p^{-i}(\Delta),n+1} - k_n\alpha_n(p) + \beta(1-p)]\psi^{-i}(\Delta) \\ &= (N-n)[V_{p^{-i},n+1} - k_n\alpha_n(p) + \beta(1-p)](\frac{\lambda p}{\alpha_n(p)})\end{aligned}$$

where the final inequality follows from the fact that at all (p,n) on-path, $\alpha_n(p) = \frac{\lambda p}{\lambda p + F'_{p,n}(0)}$.

For $K_2(\Delta)$, first note that by the continuous differentiability of $\Psi^j(s)$ that

$$\lim_{\Delta\to 0+}\sum_j \lim_{s\to\Delta-}\Psi^j(s)=0.$$

Thus, it follows from the right-differentiability of $\alpha_n(p(\Delta))$ in Δ that

$$\begin{aligned}\lim_{\Delta\to 0+}K_3(\Delta)=k_n\lim_{\Delta\to 0}\frac{\alpha_n(p(\Delta))-\alpha_n(p)}{\Delta}+\beta\lim_{\Delta\to 0+}\frac{p(\Delta)-p}{\Delta}&=k_n\frac{d}{d\Delta}\alpha_n(p(\Delta))\Big|_{\Delta=0+}+\beta p'(\Delta)\Big|_{\Delta=0+}\\ &=p'(\Delta)\Big|_{\Delta=0+}[k_n\alpha_n'(p)+\beta]=-\lambda pn(1-p)[k_n\alpha_n'(p)+\beta]\end{aligned}$$

Since we have shown that $\lim_{\Delta\to 0+}K_1(\Delta)$, $\lim_{\Delta\to 0+}K_2(\Delta)$, and $\lim_{\Delta\to 0+}K_3(\Delta)$ exist, and are given by the above expressions, it follows from (14) that

$$\lim_{\Delta\to 0+}K_1(\Delta)+\lim_{\Delta\to 0+}K_2(\Delta)+\lim_{\Delta\to 0+}K_3(\Delta)=0.$$

Substituting in the above expressions for $K_1(\Delta)$, $K_2(\Delta)$ and $K_3(\Delta)$, we obtain (ODE).

Now, we wish to establish that (ODE) must hold whenever $k_n\geq\beta$ or $p>p_n^*$. It follows from Proposition 1 that $\alpha_n(p)<1$, and thus by the above, (ODE) must hold.

Finally, we establish the two limit conditions presented in the proposition. We begin by establishing that when $k_n\geq\beta$, $\lim_{p\to 0+}\alpha_n(p)=\beta/k_n$. To this end, first note by Lemma 2 that for all $p>0$, $V_{p,n}(\delta_0)=V_{p,n}(\delta_\infty)$. Note further that

$$\lim_{p\to 0+}V_{p,n}(\delta_\infty)=0.$$

Thus,

$$\lim_{p\to 0+}V_{p,n}(\delta_0)=\lim_{p\to 0+}k_n\alpha_n(p)-\beta=0,$$

and therefore, $\lim_{p\to 0+}\alpha_n(p)=\frac{\beta}{k_n}$. Next, let us consider the case where $k_n<\beta$. That $\lim_{p\to p_n^*+}\alpha_n(p)=1$ follows from Lemma 4, since by Proposition 1, $\alpha_n(p_n^*)=1$. □

Before proceeding with the rest of the characterization, I define a problem (P) on α. I then show that α consistutes an equlibrium if and only if it satisfies (P) (Lemma 5). Thus, existence and uniqueness of an equilibrium (Theorem 1) will reduce to establishing a unique solution to (P).

Definition 2. α is a solution to (P) if it satisfies the following for all $n\leq N$ and $p\in(0,1]$:

- If $k_n<\beta$ and $p\leq p_n^*\equiv\frac{k_n-\beta}{k_n/n-\beta}$, then $\alpha_n(p)=1$.

- If $k_n \geq \beta$ or $p < p_n^*$, then α satisfies (ODE), with limit condition $\lim_{p\to 0+} \alpha_n(p) = \beta/k_n$ if $k_n \geq \beta$ and $\lim_{p\to p_n*+} \alpha_n(p) = 1$ if $k_n < \beta$.
- $\alpha_n(1) = 0$.

Lemma 5. *(α, F) is an equilibrium if and only if at all (p, n) on-path, α is both consistent with F and a solution to (P).*

Proof of Lemma 5. Fix an (α, F). I begin by establishing the necessity of the three conditions specified in Definition 2 for (α, F) to be an equilibrium. First we establish the necessity of the first bullet of Definition 2. To this end, recall that by the selection assumption, $F_{1,n}(0) = 1$. Thus, it follows from (3) that $\alpha_n(1) = 0$ if $(p = 1, n)$ is on-path. Bullets two and three of Definition (2) follow immediately from Proposition 1 and Proposition 2, respectively.

Next, we establish the sufficiency of the above conditions for (α, F) to be an equilibrium. We begin by considering the case in which $k_n < \beta$ and $p \leq p_n^*$. It follows from (P) that $\alpha_n(q) = 1$ for all $q \leq p$. Thus, by (3), $F_{p,n} = \delta_\infty$. We wish to show that there exist no profitable deviations in this case, i.e., that $V_{p,n} = V_{p,n}(\delta_\infty)$. It suffices to show that

$$V_{p,n}(\delta_\infty) \geq V_{p,n}(\delta_s) \text{ for all } s \in [0, \infty). \tag{15}$$

First, note that for all $s \in (0, \infty)$,

$$V_{p,n}(\delta_s) = k_n(1 - p(1 - e^{-\lambda sn})(\frac{N-n}{N-n+1})) - \beta(1-p) \leq k_n - \beta(1-p) = V_{p,n}(\delta_0).$$

Further, $k_n \leq \beta$ and $p \leq p_n^*$ implies that

$$V_{p,n}(\delta_0) = k_n - \beta(1-p) \leq \frac{k_n}{n} = V_{p,n}(\delta_\infty)$$

Thus, $V_{p,n}(\delta_\infty) \geq V_{p,n}(\delta_s)$ for all $s \in [0, \infty)$

Next, we show that $F_{p,n}$ is optimal when $k_n \geq \beta$ or $p < p_n^*$. To this end, we begin by showing that

$$\frac{d}{d\Delta} V_{p,n}(\delta_\Delta) = 0 \text{ for all } \Delta \in [0, \infty) \text{ if } k_n \geq \beta \text{ and for all } \Delta \in [0, t^*) \text{ if } k_n < \beta \tag{16}$$

where t^* is the unique value such that $p(t^*) = p_n^*$. Note that

$$V_{p,n}(\delta_\Delta) = \int_0^\Delta k_n\alpha_n(p(s))d\Psi^i(s) + \int_0^\Delta V_{p^i(s),n+1}d\Psi^{-i}(s) + \\ (1 - \sum_j \Psi^j(\Delta))(\alpha_n(p(\Delta)) - \beta(1 - p(\Delta))) \tag{17}$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p,n}$. Then, it follows that

$$\begin{aligned}
&\frac{d}{d\Delta}V_{p,n}(\delta_\Delta) \\
&= k_n\alpha_n(p(\Delta))\Psi^{i\prime}(\Delta) + (N-n)V_{p^i(\Delta),n+1}\Psi^{-i\prime}(\Delta) + (1 - \sum_j \Psi^j(\Delta))p'(\Delta)[\alpha_n'(p(\Delta)) - \beta] \\
&- \sum_j \Psi^{j\prime}(\Delta)(k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta))) \\
&= (N-n)[V_{p^i(\Delta),n+1} - k_n\alpha_n(p(\Delta)) + \beta(1-p(\Delta))]\Psi^{-i\prime}(\Delta) - \beta(1-p(\Delta))\Psi^{i\prime}(\Delta) \\
&+ (1 - \sum_j \Psi^j(\Delta))p'(\Delta)(k_n\alpha_n'(p(\Delta)) - \beta),
\end{aligned}$$

where $\Psi^{i\prime}(t) \equiv \frac{d}{dt}\Psi^i(t)$.

In the above, the existence of $\Psi^{j\prime}(\Delta)$ follows from the differentiability of α_n at $p(\Delta)$, and thus, the differentiability of $F_{p,n}$ at Δ. We wish to show that $\frac{d}{d\Delta}V_{p,n}(\delta_\Delta) = 0$. To this end, we begin by deriving expressions for $\Psi^{i\prime}(\Delta)$ and $\Psi^{-i\prime}(\Delta)$. First, it follows by definition of the first-report distribution that:

$$\Psi^i(\Delta) = p\lambda \int_0^\Delta (1 - F_{p,n}(s))^{N-n}e^{-\lambda ns}ds.$$

Differentiating this, we obtain:

$$\Psi^{i\prime}(\Delta) = p\lambda(1 - F_{p,n}(\Delta))^{N-n}e^{-\lambda n\Delta}$$

Meanwhile:

$$\Psi^{-i}(\Delta) = p\int_0^\Delta (1-F_{p,n}(s))^{n-2}e^{-\lambda(N-n)s}d((F_{p,n}(s)-1)e^{-\lambda s}) + (1-p)\int_0^\Delta (1-F_{p,n}(s))^{n-2}F_{p,n}'(s)ds$$

where the existence of $F_{p,n}'(s)$ again follows from the assumption that α_n is differentiable at $p(s)$. Differentiating this, we obtain:

$$\Psi^{-i\prime}(\Delta) = p(1 - F_{p,n}(\Delta))^{n-2}e^{-\lambda\Delta n}[F'_{p,n}(\Delta) + \lambda(1 - F_{p,n}(\Delta))] + (1 - p)(1 - F_{p,n}(\Delta))^{n-2}f_{p,n}(\Delta)$$
$$= (1 - F_{p,n}(\Delta))^{N-n}[\frac{f_{p,n(\Delta)}}{1 - F_{p,n}(\Delta)}(pe^{-\lambda\Delta n} + (1 - p)) + pe^{-\lambda\Delta n}\lambda]$$

It follows from (3) and (2) that

$$\frac{F'_{p,n}(\Delta)}{1 - F_{p,n}(\Delta)} = \lambda p(\Delta)(\frac{1}{\alpha_n(p(\Delta))} - 1).$$

Substituting this, along with the definition of $p(\Delta)$ (**??**), we obtain:

$$\Psi^{-i\prime}(\Delta) = \lambda(1 - F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta n} + (1 - p))\frac{p(\Delta)}{\alpha_n(p(\Delta))}$$

Note further that

$$1 - \sum_j \Psi^j(\Delta) = (1 - F(\Delta))^{N-n}(pe^{-\lambda\Delta n} + (1 - p)) \tag{18}$$

Substituting equations the expressions for $\Psi^{i\prime}(\Delta)$, $\Psi^{-i\prime}(\Delta)$, and $1 - \sum_j \Psi^j(\Delta)$ into the above equation for $\frac{d}{d\Delta}V_{p,n}(\delta_\Delta)$, and simplifying, we obtain:

$$\frac{d}{d\Delta}V_{p,n}(\delta_\Delta) = K[\frac{(N - n)}{\alpha_n(p(\Delta))}(V^i_{p(\Delta),n+1} - k_n\alpha_n(p(\Delta)) + \beta(1 - p(\Delta))(1 - \alpha_n(p(\Delta))))$$
$$-k_n\alpha'_n(p(\Delta))(1 - p(\Delta))n]$$

where $K \equiv \lambda(1-F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta n}+(1-p))p(\Delta)$. Because (ODE) is satisfied at $(p(\Delta), n)$, using it to substitute in for $\alpha'_n(p(\Delta)$, we obtain (16).

Now, consider the case where $k_n \geq \beta$. To show $F_{p,n}$ is optimal, it suffices to show that all pure strategies δ_Δ yield the same payoff, i.e., that

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_\Delta) \tag{19}$$

for all $\Delta \in [0, \infty]$. It follows directly from (16) that (19) holds for all $\Delta \in [0, \infty)$. It remains to show that (19) holds for $\Delta = \infty$. To this end, first note that by (16),

$$\begin{aligned}
&V_{p,n}(\delta_0) = \lim_{\Delta\to\infty} V_{p,n}(\delta_\Delta) \\
&= \lim_{\Delta\to\infty} \int_0^\Delta k_n\alpha_n(p(s))d\Psi^i(s) + (N-n)\lim_{\Delta\to\infty}\int_0^\Delta V_{p^i(s),n+1}d\Psi^{-i}(s)+ \\
&\lim_{\Delta\to\infty}(1-\sum_j \Psi^j(\Delta))(k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta))) \\
&= \int_0^\infty k_n\alpha_n(p(\Delta))d\Psi^i(s) + (N-n)\int_0^\infty V_{p^i(\Delta),n+1}d\Psi^{-i}(s) = V_{p,n}(\delta_\infty)
\end{aligned}$$

where the third equality follows from the limit condition $\lim_{p\to 0+}\alpha_n(p) = \beta/k_n$:

$$\lim_{\Delta\to\infty} k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta)) = \lim_{p\to 0+} k_n\alpha_n(0) - \beta = 0.$$

Finally, consider the case where $k_n < \beta$ and $p > p_n^*$. Because $\alpha_n(p(s)) = 1$ for all $s > t^*$, by (3), it follows that $F'_{p,n}(s) = 0$ for all such s. Then, the support of $F_{p,n}$ lies within $[0,t^*] \cup \infty$. Thus, to show $F_{p,n}$ is optimal, it suffices to show that δ_Δ is optimal for $\Delta \in [0,t^*]\cup\infty$. To this end, I first show that

$$V_{p,n}(\delta_\Delta) = V_{p,n}(0) \text{ for all } \Delta \in [0,t^*]\cup\infty \tag{20}$$

and then show

$$V_{p,n}(\delta_{t^*}) \geq V_{p,n}(\delta_\Delta) \text{for all } \Delta \in (t^*,\infty). \tag{21}$$

To show (20), first recall that it follows from (16) that

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_\Delta) \text{ for all } \Delta \in [0,t^*).$$

It remains to show $V_{p,n}(\delta_0) = V_{p,n}(\delta_s)$ for $s \in \{t^*,\infty\}$. For $s = t^*$, it follows from the above that

$$V_{p,n}(\delta_0) = \lim_{\Delta\to t^*-} V_{p,n}(\delta_\Delta) = V_{p,n}(\delta_{t^*})$$

where the final inequality follows from (17), observing that α_n is continuous at p_n^* and Ψ^j is continuous at t^*. I will now show $V_{p,n}(\delta_{t^*}) = V_{p,n}(\delta_\infty)$. To this end, note that for all $\Delta \in [t^*,\infty]$:

$$V_{p,n}(\delta_\Delta) = \int_0^{t^*} k_n\alpha_n(p(s))d\Psi^i(s) + (N-n)\int_0^{t^*} V_{p^i(s),n+1}d\Psi^{-i}(s) + (1-\sum_j \Psi^j(t^*))V_{p_n^*,n}(\delta_{\Delta-t^*})$$

Thus, to show $V_{p,n}(\delta_t^*) = V_{p,n}(\delta_\infty)$, it suffices to show that $V_{p_n^*,n}(\delta_0) = V_{p_n^*,n}(\delta_\infty)$. But it follows from the definition of p_n^* that:

$$V_{p_n^*,n}(\delta_0) = k_n - \beta(1-p_n^*) = \frac{k_n p_n^*}{n} = V_{p_n^*,n}(\delta_\infty).$$

Similarly, to show (21), it suffices to show that $V_{p_n^*,n}(\delta_0) \geq V_{p_n^*,n}(\delta_\Delta)$ for all $\Delta \in (0,\infty)$, which we have established in (15). □

Proof of Theorem 1. Fix an n. Assume by induction that there exists a unique solution to (P) for all $m > n$. We wish to show that there exists a unique solution to (P) for n. To establish this, it suffices to show there exists a unique solution to the following two limit problems, when $\beta \leq k_n$ and $\beta > k_n$, respectively:

$$\text{(ODE) is satisfied on } (0,1)\text{, and } \lim_{p\to 0+} \alpha_n(p) = \beta/k_n \qquad \text{(LP: } \beta \leq k_n\text{)}$$

$$\text{(ODE) is satisfied on } (0,p^*)\text{, and } \lim_{p\to p_n^*+} \alpha_n(p) = 1. \qquad \text{(LP: } \beta > k_n\text{)}$$

To establish existence and uniqueness to the two above problems, we proceed by extending them to two boundary value problems. To this end, we begin by defining an extension of (ODE′) of (ODE), which is identical to (ODE), except that it is well-defined when $p^i \geq 1$. Specifically, define:

$$\alpha_n'(p) = -\frac{1}{k_n(1-p)\alpha_n(p)}\frac{N-n}{N-n+1}[k_n\alpha_n(p) - \tilde{V}_{p^i,n+1} - \beta(1-\alpha_n(p))(1-p)] \qquad \text{(ODE′)}$$

where

$$\tilde{V}_{p^i,n+1} = \begin{cases} V_{p^i,n+1} & \text{if } p^i \in (0,1) \\ 0 & \text{if } p^i \geq 1 \end{cases}$$

Now let us define two boundary value problems on (ODE′):

$$\text{(ODE′) is satisfied on } [0,1)\text{, and } \alpha_n(0) = \beta/k_n \qquad \text{(BVP: } \beta \leq k_n\text{)}$$

$$\text{(ODE′) is satisfied on } (0,p_n^*]\text{, and } \alpha_n(p^*) = 1. \qquad \text{(BVP: } \beta \geq k_n\text{)}$$

Now we claim that the existence and uniqueness of a solution to (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$) implies the existence and uniqueness of a solution to (LP: $\beta \leq k_n$) and (LP: $\beta > k_n$), respectively. Let us begin by considering the case where $k_n \geq \beta$. Assume that there exists a unique solution α_n to (BVP: $\beta \leq k_n$). Note that order for α_n to satisfy (BVP: $\beta \leq k_n$), it must be that $\lim_{p\to 0+} \alpha_n(p) = k_n/\beta$. Furthermore, (ODE) and (ODE′) are equivalent on $(0,1)$. It follows that α_n is a solution to (LP: $\beta \leq k_n$), thus establishing existence. To establish uniqueness, assume by contradiction there exists some $\tilde{\alpha}_n$ defined on $p \in (0,1)$ that is a solution to (LP: $\beta \leq k_n$) where $\tilde{\alpha}_n(p) \neq \alpha_n(p)$. Now, define $\hat{\alpha}_n$, which extends the domain of $\tilde{\alpha}_n$, as follows:

$$\hat{\alpha}_n(p) = \begin{cases} \tilde{\alpha}_n(p) & \text{if } p \in (0,1) \\ k_n/\beta & \text{if } p = 0 \end{cases}$$

Because $\lim_{p\to 0+} \tilde{\alpha}_n(p) = k_n/\beta$, it follows that $\hat{\alpha}_n(p)$ satisfies (ODE′) on $p \in [0,1]$ and is thus a solution to (BVP: $\beta \leq k_n$). Thus, (BVP: $\beta \leq k_n$) does not have a unique solution, a contradiction. Note that the argument in the case where $k_n < \beta$ is analogous.

It remains for us to establish that there exist unique solutions to both (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$). We do this by invoking the Picard existence and uniqueness theorem, and thus begin by establishing that the right-hand side of (ODE′) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p for $p \in [-\varepsilon, 1)$ and $\alpha_n(p) \in [c, 1+\varepsilon]$ for any $c > 0$ and some $\varepsilon > 0$. Since $p^i \equiv \alpha_n(p) + (1-\alpha_n(p))p$, it suffices to show that $\tilde{V}_{\cdot,n+1}$ is Lipschitz continuous in p^i for $p^i \geq 0$. In the case where $n = 1$, $\tilde{V}_{p^i,n+1} = 0$ for all p^i, and this is immediate. Next, suppose $n > 1$. First, consider the case where $k_{n+1} \geq \beta$. It follows from Lemma 2 that:

$$\tilde{V}_{p^i,n+1} = \begin{cases} k_n\alpha_{n+1}(p^i) - \beta(1-p^i) & \text{if } p^i < 1 \\ 0 & \text{if } p^i > 1 \end{cases}$$

Because $\tilde{V}_{p^i,n+1}$ is continuously differentiable in p^i when $p^i \neq 1$, to establish that it is Lipschitz continuous it suffices to show that $\lim_{p^i\to 1-} V_{p^i,n+1} = 0$. Suppose this does not hold, by contradiction. Because $\alpha_{n+1}(\cdot)$ satisfies (ODE), this implies that $\lim_{p^i\to 1-} \alpha'_{n+1}(p^i) = \infty$. This in turn implies that $\lim_{p^i\to 1} \alpha_{n+1}(p^i) = \infty$, and thus that (ODE) is not satisfied at $p^i = 1$. Contradiction.

Next, consider the case where $k_{n+1} < \beta$. In this case:

$$\tilde{V}_{p^i,n+1} = \begin{cases} k_{n+1}p^i/n & \text{if } p^i < p^*_{n+1} \\ k_n\alpha_{n+1}(p^i) - \beta(1-p^i) & \text{if } p^i \in (p^*_{n+1}, 1) \\ 0 & \text{if } p^i = 1 \end{cases}$$

By the reasoning from the case where $k_{n+1} \geq \beta$, $\tilde{V}_{p^i,n+1}$ is Lipschitz continuous for all $p^i > p^*_{n+1}$. Furthermore, Lipschitz continuity holds on $p^i < p^*_{n+1}$. To show that Lipschitz continuity holds across all p^i, it suffices to show that $\tilde{V}_{\cdot,n+1}$ is differentiable at p^*_{n+1}. To this end, we take the left- and right- derivative of $\tilde{V}_{\cdot,n+1}$ at p^*_{n+1} and show that they are equal:

$$\frac{d}{dp}\tilde{V}_{p^*-,n+1} = \frac{k_{n+1}}{N-n}$$

$$\frac{d}{dp}\tilde{V}_{p^*+,n+1} = -k_{n+1}\alpha'_{n+1}(p^*_{n+1}) + \beta = \frac{k_{n+1}}{1-p^*_{n+1}}\frac{N-n}{N-n+1} - \beta = \frac{k_{n+1}}{N-n}$$

Now, we show that there exists a unique solution for both (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$) in some neighborhood of their respective boundary conditions. By the Picard Theorem, this follows immediately from our above-established result that the right-hand side of (ODE) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p in some neighborhood of the boundary conditions $(\alpha_n(p) = 1, p = p^*)$ and $(\alpha_n(p) = \beta/k_n, p = 0)$.

Next, we seek to establish global existence and uniqueness of solutions to both (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$). First, consider (BVP: $\beta \geq k_n$). The argument for (BVP: $\beta \leq k_n$) follows analogously. Let $[p^*, \overline{p})$ denote the largest right-open interval such that existence and uniqueness are both satisfied. Assume by contradiction that $\overline{p} < 1$. Let $\alpha_n(p)$ denote the solution along this interval.

We begin by showing that on this interval, $\alpha_n(p) \in (\underline{\alpha}, 1]$, where $\underline{\alpha} > 0$ is some constant. The upper bound is established as follows: suppose by contradiction that $\alpha_n(p) > 1$ somewhere on the interval. By the continuous differentiability of α_n along the interval, there must exist some $q < p$ such that $\alpha_n(q) = 1$ and $\alpha'_n(q) \geq 0$. However, it follows from (ODE′) that

$$\alpha'_n(q) = -\frac{1}{k_n(1-q)}\frac{N-n}{N-n+1}[k_n - \tilde{V}_{p^i,n+1}] < 0$$

where the strict inequality follows from the fact that $\tilde{V}_{p^i,n+1} \leq k_{n+1} < k_n$. Contradiction. The lower bound is established as follows: suppose by contradiction that such a lower bound does not exist. Then, again by the continuous differentiability of α_n along the interval, there exists some $\hat{p} \in [p^*, \overline{p})$ such that

$$\lim_{p \to \hat{p}-} \alpha_n(p) = 0 \text{ and } \alpha_n(p) > 0 \text{ for all } p < \hat{p}$$

However, it then follows from (ODE) that $\lim_{p\to\hat{p}-} \alpha'_n(p) = \infty$. Thus, (ODE′) is not satisfied on $[p^*, \overline{p})$. Contradiction.

Having established that on $[p^*, \overline{p})$, $1 \leq \alpha_n(p) > \underline{\alpha} > 0$, it follows from (ODE′), and the observation that $V_{p^i,n+1}$ is bounded, that α'_n is also bounded on this range. Thus, it follows that $\lim_{p\to\overline{p}-} \alpha_n(p) \equiv \overline{\alpha} > 0$ exists.

Now, consider the following modified boundary value problem, which is identical to (BVP: $\beta \geq k_n$), except with boundary condition $(\overline{p}, \overline{\alpha})$. by our prior-established result, we recall that (ODE′) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p in some neighborhood of the boundary condition. Thus, we can again apply the Picard Theorem to obtain that there exists a unique solution to the modified boundary value problem in some neighborhood of $(\overline{p}, \overline{\alpha})$. I.e., there exists some $\varepsilon > 0$ such that there is a unique solution $\tilde{\alpha}_n(p)$ on interval $(\overline{p} - \varepsilon, \overline{p} + \varepsilon)$. We can "paste" this solution $\tilde{\alpha}_n$, with our prior solution α_n. Let

$$\hat{\alpha}_n(p) = \begin{cases} \alpha_n(p) & \text{if } p \in [p_n^*, \overline{p}) \\ \tilde{\alpha}_n(p) & \text{if } p \in [\overline{p}, \overline{p} + \varepsilon) \end{cases}$$

Now, note that $\hat{\alpha}_n(p)$ is a unique solution to (BVP: $\beta \geq k_n$) on $[p_n^*, \overline{p}+\varepsilon)$, which contradicts our earlier assumption that $[p^*, \overline{p})$ was the largest right-open interval such that existence and uniqueness are satisfied. Contradiction. □

Proof of Proposition 3 and ??. Let us begin by showing that $\alpha_n(p)$ is decreasing in p for all (p, n) on-path. By Lemma 5, it follows that when $k_N < \beta$, $\alpha_N(p) = 1$ for all p, and otherwise, $\alpha'_N(p) = 0$ for all p. Thus we have shown that $\alpha_N(p)$ is constant in p. Now, consider the case where $n < N$. Assume by induction that $\alpha_{n+1}(p)$ is weakly decreasing in p whenever $(p, n+1)$ is on path.

Assume by contradiction that there exists some $\overline{p}$ such that α_n is strictly increasing. Note that Lemma 5, $\alpha'_n(p) = 0$ whenever $\beta \geq k_n$ and $p < p_n^*$. Thus it must be that $\beta > k_n$ or $\overline{p} > p_n^*$. In this case, it again follows from Lemma 5 that (ODE) must be satisfied. Now define the function $X(p)$ as follows:

$$X(p) \equiv k_n \alpha_n(p) - \beta(1 - p^i) - V_{p^i, n+1} \tag{22}$$

Note that whenever (ODE) is satisfied, the following holds:

$$\alpha'_n(p) > (=)0 \text{ if and only if } X(p) < (=)0 \tag{23}$$

Thus, $X(\overline{p}) < 0$. Now, I claim that there exists $\underline{p} < \overline{p}$ such that $\lim_{p \to \underline{p}+} X(p) \geq 0$. To establish this, first consider the case where $k_n \geq \beta$. In this case,

$$\lim_{p\to 0+} X(p) = k_n \lim_{p\to 0+} \alpha_n(p) - \beta(1 - \lim_{p\to 0+}) - \lim_{p\to 0+} V_{\alpha_n(p), n+1} = (k_n + \beta) \lim_{p\to 0+} \alpha_n(p) - \beta - \lim_{p\to 0+} V_{\alpha_n(p), n+1}$$

When $\lim_{p\to 0+} \alpha_{n+1}(\alpha_n(p)) < 1$, it follows from Lemma 2 that

$$\lim_{p\to 0+} V_{\alpha_n(p),n+1} = \lim_{p\to 0+} V_{\alpha_n(p),n+1}(\delta_0) = k_{n+1} \lim_{p\to 0+} \alpha_{n+1}(\alpha_n(p)) - \beta(1 - \lim_{p\to 0+} \alpha_n(p))$$
$$= k_{n+1}\alpha_{n+1}(\beta/k_n) - \beta(1 - \beta/k_n)$$

Because $k_n \geq \beta$, the final equality follows from Lemma 5. Substituting this into our above expression for $\lim_{p\to 0+} X(p)$, we obtain

$$\lim_{p\to 0+} X(p) = \beta - k_{n+1}\alpha_{n+1}(\beta/k_n)$$

In the case where $k_{n+1} < \beta$, it follows directly that $\lim_{p\to 0+} X(p) > 0$. Otherwise, if $k_{n+1} \geq \beta$, then because $\lim_{p\to 0+} \alpha_{n+1}(p) = \beta/k_{n+1}$, it follows from the inductive assumption that $\alpha_{n+1}(p) \leq \beta/k_{n+1}$ for all p, and thus that $\lim_{p\to 0+} X(p) > 0$.

Meanwhile, when $\lim_{p\to 0+} \alpha_{n+1}(\alpha_n(p)) = 1$, it follows from the inductive assumption that $\alpha_{n+1}(q) = 1$ for all $q \geq \lim_{p\to 0+} \alpha_n(p)$. It thus follows that

$$\lim_{p\to 0+} V_{p^i,n+1} = \lim_{p\to 0+} V_{p^i,n+1}(\delta_\infty) = \frac{k_{n+1}}{N-n}\frac{\beta}{k_n}$$

Substituting into the above expression for $\lim_{p\to 0+} X(p)$ and simplifying, we obtain

$$\lim_{p\to 0+} X(p) = (\beta/k_n)(\beta - k_{n+1}/(N-n)) \geq 0,$$

where the inequality follows from the fact that $\alpha_{n+1}(\beta/k_n) = 1$, implying by Lemma 5 that $k_{n+1} \geq \beta$.

Next, consider the case where $k_n < \beta$. In this case,

$$\lim_{p\to p_n^*+} X(p) = k_n - \lim_{p^i\to 1-} V_{p^i,n+1}$$

If $\lim_{p^i\to 1-} \alpha_{n+1}(p^i) < 1$, then by Lemma 2,

$$\lim_{p^i\to 1-} V_{p^i,n+1} = \lim_{p^i\to 1-} V_{p^i,n+1}(\delta_0) = k_{n+1} \lim_{p^i\to 1-} \alpha_{n+1}(p^i) < k_n.$$

Thus, in this case, we obtain that $\lim_{p\to p_n^*+} X(p) > 0$. Meanwhile, if $\lim_{p^i\to 1-} \alpha_{n+1}(p^i) = 1$, by the inductive assumption, $\alpha_{n+1}(p) = 1$ for all p. Thus,

$$\lim_{p^i\to 1-} V_{p^i,n+1} = \lim_{p^i\to 1-} V_{p^i,n+1}(\delta_\infty) = \lim_{p^i\to 1-} \frac{k_{n+1}p^i}{N-n} = \frac{k_{n+1}}{N-n}.$$

We once again obtain $\lim_{p\to p_n^*+} X(p) > 0$. We have thus shown that there always exists $\underline{p} < \overline{p}$ such that $\lim_{p\to \underline{p}+} X(p) \geq 0$.

Because $X(\overline{p}) < 0$ by assumption, there must exist some $q \in [\underline{p}, \overline{p}]$ $X(q) < 0$ and $X'(q) < 0$. Note that differentiating our above expression for X, we have

$$X'(q) = k_n \alpha_n'(q) + \beta((1-q)\alpha_n'(q) + (1-\alpha_n(q))) - \frac{d}{dq} V_{\alpha_n(q)+(1-\alpha_n(q))q, n+1}. \tag{24}$$

First, consider the case where $\alpha_{n+1}(q^i) < 1$. By Lemma 2,

$$V_{q^i,n+1} = V_{q^i,n+1}(\delta_0) = k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i).$$

Substituting this into (24), we obtain

$$X'(q) = k_n \alpha_n'(q) - k_{n+1}\alpha_{n+1}'(q^i)((1-q)\alpha_n'(q) + (1-\alpha_n(q))).$$

Note that because $X(q) < 0$ it follows from (23) that $\alpha_n'(q) > 0$. Furthermore, by the inductive assumption, $\alpha_{n+1}'(q^i) \leq 0$. Thus, in this case, $X'(q) > 0$. Contradiction.

Next, consider the case where $\alpha_{n+1}(q^i) = 1$. By the inductive assumption, $\alpha_{n+1}(p) = 1$ for all $p \leq q^i$. Thus,

$$V_{q^i,n+1} = V_{q^i,n+1}(\delta_\infty) = \frac{k_{n+1}q^i}{N-n}.$$

Substituting this into (24), we obtain

$$X'(q) = k_n \alpha_n'(q) + (\beta - \frac{k_{n+1}}{N-n})((1-q)\alpha_n'(q) + (1-\alpha_n(q)))$$

Because $\alpha_{n+1}(q^i) = 1$, by Proposition 1 (if $n < N-1$) and Lemma 5 (if $n = N-1$), it must be that $\beta \geq k_{n+1}$. Thus, $X'(q) > 0$. Contradiction.

Next, we will show that if $k_N \geq \beta$, then $\alpha_n(p) = \beta/k_n$. Assume that $k_N \geq \beta$. First consider the case where $n = N$. By Lemma 5, $\alpha_n'(p) = 0$ for all p on-path, and thus, $\alpha_N(p)$ is constant in p. Since Lemma 5 also asserts that $\lim_{p\to 0+} k_N\alpha_N(p) = \beta$, it must be that $\alpha_N(p) = \beta/k_N$ for all p. Now, consider $n < N$. Assume by induction that $\alpha_{n+1}(p) = \beta/k_{n+1}$ for all p. We begin by showing that $\alpha_n(p)$ is constant in p. Since $k_n > \beta$, by Lemma 5, (ODE) must hold at all p. By (23), showing $\alpha_n(p)$ is constant in p is equivalent to showing that $X(p) = 0$. To establish this, I begin by claiming that $V_{p^i,n+1} = V_{p^i,n+1}(\delta_0)$. In the case where $k_{n+1} > \beta$, it follows from Proposition 1 that $\alpha_{n+1}(p^i) < 1$, and thus this follows from Lemma 2. In the case where $k_{n+1} = \beta$, because $k_m > k_N \geq \beta$ for all $m < N$, it follows that $n+1 = N$. In this case, all pure strategies δ_s must yield the same value. In particular, for all

$s \in [0, \infty]$, $V_{p,N}(\delta_s) = k_N p$. Thus, δ_0 is optimal. Having established that $V_{p^i,n+1} = V_{p^i,n+1}(\delta_0)$, we have:

$$V_{p^i,n+1} = k_{n+1}\alpha_{n+1}(p^i) - \beta(1 - p^i) = \beta p^i$$

Substituting this into (22), we obtain $X(p) = k_n \alpha_n(p) - \beta$. Since we established above that $\alpha_n(p)$ is weakly decreasing, $\alpha_n(p) \leq k_n/\beta$ for all p, and thus $X(p) \leq 0$. Separately, by (23) $\alpha_n(p)$ weakly decreasing implies that $X(p) \geq 0$. Combining these inequalities, we have $X(p) = 0$.

Finally, I will show that $k_N < \beta$ implies that $\alpha_n'(p) < 0$ whenever $\alpha_n(p) < 1$. To this end, suppose $k_N < \beta$, and suppose by contradiction that at some q such that $\alpha_n(q) < 1$, $\alpha_n'(q) = 0$. It follows from (23) that $X(q) = 0$.

First, consider the case where $\alpha_{n+1}(q^i) = 1$. Recall from the above that in this case, we have

$$X'(q) = k_n \alpha_n'(q) + (\beta - \frac{k_{n+1}}{N-n})((1-q)\alpha_n'(q) + (1 - \alpha_n(q)) = (\beta - \frac{k_{n+1}}{N-n})(1 - \alpha_n(q)) \quad (25)$$

Now, I claim that $\beta > \frac{k_{n+1}}{N-n}$. In the case where $n = N - 1$, this follows directly from our assumption that $k_N < \beta$. Meanwhile, in the case where $n < N - 1$, because $\alpha_{n+1}(q^i) = 1$, this is a result of Proposition 1. It thus follows from (25) that $X'(q) > 0$. Since $X(q) = 0$, for some $p < q$, we must have $X(p) < 0$. By (24), $\alpha_n'(p) > 0$. This contradicts the above-established assertion that $\alpha_n(p)$ is weakly decreasing in p.

Next, consider the case where $\alpha_{n+1}(q^i) < 1$. As established above, in this case:

$$X'(q) = k_n\alpha_n'(q) - k_{n+1}\alpha_{n+1}'(q^i)[(1-q)\alpha_n'(q) + (1 - \alpha_n(q))] = -k_{n+1}\alpha_{n+1}'(q)[1 - \alpha_n(q)] > 0.$$

Again, this implies that there exists some $p < q$ such that $X(p) < 0$ and thus that $\alpha'(p) > 0$. Contradiction. $\square$

Appendix D Commitment solution

Here, we seek the optimal solution to the monopoly case of the baseline model in which the firm has the ability to commit to a reporting strategy. The only modification we introduce is that rather than F and α being determined simultaneously as they are in equilibrium, the firm chooses its strategy F before α is determined. Thus, in the commitment case, the credibility function is a function of the firm's strategy. We formalize this dependence by denoting the firm's credibility function as α_F. α_F is then given by (3) as in the non-commitment case, except that the strategy F upon which it is computed is the firm's choice of strategy, rather than the equilibrium strategy.

Because we are considering the monopoly case only, I will be dropping the n index from all functions and expressions. Furthermore, for convenience, I will be writing all functions as a function of calendar time t, rather than the common belief p as in the baseline model. Writing the functions in this way is without loss, since under a monopoly there is a one-to-one correspondence between the calendar time t and the common belief p.

The firm's objective is to choose a permissible strategy $F \in \mathcal{F}$ which maximizes its expected payoff over the course of the game. Specifically, its problem is given by the following:

$$\max_{F\in\mathcal{F}} \int_0^\infty [\alpha_F(t) - \beta(1-p(t))(1-\alpha_F(t))]d\Psi(t) \tag{26}$$

where, as in the baseline setup, $\Psi(t)$ denotes probability that the firm reports before time t under strategy F. Before proceeding, we highlight that the only difference between this problem and the problem of the monopoly case of the baseline model is that the credibility function is not taken as given, but is rather a function of the firm's choice of strategy F.

In the analysis that follows, it will be useful for us to cast this problem as a choice of an optimal credibility function α, rather than an optimal strategy F. To this end, I begin with a useful observation, which is analogous to Lemma 1, except under the commitment setting:

Lemma 6. *F must be continuous in equilibrium.*

We omit a proof for this claim, as it follows analogously to the proof for Lemma 1: if F exhibits a discontinuity at some time t, reporting at this time must yield a negative expected payoff. Thus, the firm can profitably deviate by shifting the mass that it had placed on reporting t to δ_∞.

It follows immediately from Lemma 6 that in equilibrium, both the firm's strategy F and the corresponding commitment function, α_F, are defined by the right-hazard rate $b(t)$ of the firm's strategy. That is,

$$\alpha_F(t) = \frac{\lambda p(t)}{\lambda p(t) + b(t)}$$

It further follows that Ψ is continuous and can thus be written as a function of α_F as follows:

$$\Psi(t) = 1 - e^{-\int_0^t (b(s)+p(s)\lambda)ds} = 1 - e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}$$

Having written Ψ in terms of α_F, and noting that at any given t $\alpha_F(t)$ is a one-to-one function of $b(t)$, we can cast the optimization problem given by (26) as one over α_F:

$$\max_{\alpha_F} \int_0^\infty \lambda p(t)[1-\beta(1-p(t))(\frac{1}{\alpha_F(t)}-1)]e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}$$

In the following claim, I show that the optimal strategy for the firm consists of always truth telling (i.e., $\alpha_F(t) = 1$ for all t). In the proof that follows, I let $V(t, \alpha_F)$ denote the firm's value at time t given that it has chosen α_F.

Proposition 7. *In equilibrium, $\alpha_F(t) = 1$ for all t.*

Proof. Assume not, by contradiction. Then there exists a t^* such that $\alpha_F(t^*) < 1$. It follows from Lemma 6, and the assumption that F is right-continuously differentiable, that α_F must be right-continuous. Thus, there must exist a $\hat{\alpha} < 1$ and $\varepsilon > 0$ such that $\alpha_F(t) < \hat{\alpha}$ for all $t \in [t^*, t^* + \varepsilon]$.

Note that for any α_F, including the equilibrium α_F, we can write the time-0 value as follows:

$$V(0,\alpha_F) = \int_0^{t^*+\varepsilon} \lambda p(t)[1-\beta(1-p(t))](\frac{1}{\alpha_F(t)}-1)e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}dt + e^{-\int_0^{t^*+\varepsilon} \frac{\lambda p(s)}{\alpha_F(s)}ds} V(t^*+\varepsilon, \alpha_F) \tag{27}$$

Now, consider the following deviation $\tilde{\alpha}_F$, which is identical to α_F, except that it is 1 on the interval $[t^*, t^* + \varepsilon]$:

$$\tilde{\alpha}_F(t) = \begin{cases} 1 & \text{if } t \in [t^*, t^*+\varepsilon] \\ \alpha_F(t) & \text{otherwise} \end{cases}$$

Now, it follows from (27) that

$$\begin{aligned} V(0,\alpha_F) = V(0,\tilde{\alpha_F}) + \int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1-\beta(1-p(t))(\frac{1}{\alpha_F(t)}-1)]e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}dt - \int_{t^*}^{t^*+\varepsilon} \lambda p(t) e^{-\int_0^t \lambda p(s)ds}dt \\ +(e^{-\int_{t^*}^{t^*+\varepsilon} \frac{\lambda p(s)}{\alpha_F(s)}ds} - e^{-\int_{t^*}^{t^*+\varepsilon} \lambda p(s)ds})V(t^*+\varepsilon, \alpha_F) \end{aligned} \tag{28}$$

Now, we will note the following two inequalities:

$$\begin{aligned} \int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1-\beta(1-p(t))(\frac{1}{\alpha_F(t)}-1)]e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}dt \le \int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1-\beta(1-p(t))(\frac{1}{\overline{\alpha}}-1)]e^{-\int_0^t \frac{\lambda p(s)}{\overline{\alpha}}ds}dt \\ < \int_{t^*}^{t^*+\varepsilon} \lambda p(t) e^{-\int_0^t \lambda p(s)ds}dt \end{aligned}$$

$$e^{-\int_{t^*}^{t^*+\varepsilon}\frac{\lambda p(s)}{\alpha_F(s)}ds} - e^{-\int_{t^*}^{t^*+\varepsilon}\lambda p(s)ds} \leq e^{-\int_{t^*}^{t^*+\varepsilon}\frac{\lambda p(s)}{\tilde{\alpha}}ds} - e^{-\int_{t^*}^{t^*+\varepsilon}\lambda p(s)ds} < 0$$

Applying these two inequalites to (28) we obtain

$$V(0, \alpha_F) < V(0, \tilde{\alpha}),$$

and thus, $\tilde{\alpha}_F$ serves as a profitable deviation. Contradiction. □

Appendix E Proofs: comparative statics

Proof of Proposition 5. First, we establish part (a). Fix all other parameters and let $0 < \beta < \tilde{\beta}$. Let α and $\tilde{\alpha}$ denote the equilibrium credibility functions under β and $\tilde{\beta}$, respectively. Fix an n and assume inductively that the proposition holds for $n+1$ if $n < N$. Note that for any (p, n) and t, $p(t)$ will be the same under β and $\tilde{\beta}$. Thus to show the above claim, it suffices to show that for any p, $\alpha_n(p)$ is weakly increasing in β, and strictly so whenever $\alpha_n(p) < 1$.

We begin by showing that $\alpha_n(p) = 1$ implies that $\tilde{\alpha}_n(p) = 1$. First, consider the case where $n = N$. By Proposition 2, $\alpha_N(p) = 1$ implies that $k_N \leq \beta$. Thus, $k_N < \tilde{\beta}$, which by Proposition 1 implies that $\tilde{\alpha}_N(p) = 1$. Next, consider the case where $n < N$, and assume $\alpha_n(p) = 1$. By Proposition 1, this implies that $k_n < \beta$ and $p \leq p_n^* \equiv \frac{\beta - k_n}{\beta - k_n/n}$. Further note that

$$\tilde{p}_n^* \equiv \frac{\tilde{\beta} - k_n}{\tilde{\beta} - k_n/n} > \frac{\beta - k_n}{\beta - k_n/n} \equiv p_n^*.$$

Thus, $k_n < \tilde{\beta}$ and $p < \tilde{p}_n^*$, which by Proposition 1 implies $\tilde{\alpha}_n(p) = 1$.

Now, suppose that $\alpha_n(p) < 1$. We wish to show that $\tilde{\alpha}_n(p) > \alpha_n(p)$. Suppose by contradiction that $\tilde{\alpha}_n(p) \leq \alpha_n(p)$. It follows from Proposition 2 that if $k_n > \tilde{\beta}$,

$$\lim_{q\to 0+} \alpha_n(q) = \beta/k_n < \tilde{\beta}/k_n = \lim_{q\to 0+} \tilde{\alpha}_n(q).$$

Meanwhile, if $k_n \leq \tilde{\beta}$.

$$\lim_{q\to\tilde{p}_n^*+} \alpha_n(q) < 1 = \lim_{q\to\tilde{p}_n^*+} \tilde{\alpha}_n(q)$$

To see why the latter must must hold, first consider the case where $n = 1$. It follows from Lemma 5 that $\tilde{\alpha}_n(q) = 1$ for all q. Meanwhile, it follows again from Proposition 2 that $\alpha_N(q)$ is constant in q, and because $\alpha_N(p) < 1$, $\lim_{q\to\tilde{p}_n^*+} \alpha_N(q) < 1$. In the case where $n < N$, because $p_n^* < \tilde{p}_n^*$, it follows from Proposition 1 that $\alpha_n(\tilde{p}_n^*) < 1$.

Thus, we have that both when $k_n > \tilde{\beta}$ and when $k_n \leq \tilde{\beta}$, there exists some $\hat{p} < p$ such that $\tilde{\alpha}_n(\hat{p}) > \alpha_n(\hat{p})$ and $\tilde{\alpha}_n$, α_n satisfy (ODE) on $[\hat{p}, p]$, for their respective value of β. Thus, there exists a $q \in [\hat{p}, p]$ such that $\alpha_n(q) = \tilde{\alpha}_n(q)$ and $\alpha_n'(q) \geq \tilde{\alpha}_n'(q)$. It follows from (ODE) that in order for the above two conditions to hold, it must be that

$$X \equiv (\beta - \tilde{\beta})(\frac{1-\alpha_n(q)}{\alpha_n(q)})(1-q) + \frac{V_{q^i,n+1} - \tilde{V}_{q^i,n+1}}{\alpha_n(q)} \geq 0 \tag{29}$$

where V and $\tilde{V}$ denote the value functions under β and $\tilde{\beta}$, respectively. First consider the case where $n = N$. Then $V_{q^i,n+1} = V_{\tilde{q}^i,n+1} = 0$, and thus $X < 0$, contradicting (29).

Next, consider the case where $n < N$. First suppose that $\alpha_{n+1}(q^i) = 1$. It follows from the inductive assumption that $\tilde{\alpha}_{n+1}(q^i) = 1$. Thus, by Lemma 5, $V_{q^i,n+1} = \frac{k_{n+1}q^i}{N-n} = \tilde{V}_{q^i,n+1}$. Again this implies that $X < 0$, contradicting (29). Now, suppose that $\alpha_{n+1}(q^i) < 1$. It then follows from Lemma 2 that

$$V_{q^i,n+1} = V_{q^i,n+1}(\delta_0) = k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i)$$

Furthermore,

$$\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_0) = k_{n+1}\tilde{\alpha}_{n+1}(q^i) - \tilde{\beta}(1-q^i)$$

Thus, recalling from (**??**) that $q^i = \alpha_{n+1}(q) + (1-\alpha_{n+1}(q))q$, we have

$$V_{q^i,n+1} - \tilde{V}_{q^i,n+1} \leq k_{n+1}(\alpha_{n+1}(q^i) - \tilde{\alpha}_{n+1}(q^i))$$

Substituting this into the above expression for X, we obtain

$$X \leq \frac{k_{n+1}(\alpha_{n+1}(q^i) - \tilde{\alpha}_{n+1}(q^i))}{\alpha_n(q)} < 0.$$

where the strict inequality follows from the inductive assumption that $\alpha_{n+1}(q^i) < \tilde{\alpha}_{n+1}(q^i))$. Again, this is a contradiction of (29).

Next, let us establish part (b). Let $\tilde{\lambda} > \lambda > 0$, and let α, $\tilde{\alpha}$ denote the equilibria under λ and $\tilde{\lambda}$, respectively, fixing all other parameters. We begin by showing that $\tilde{\alpha}_n(p) = \tilde{\alpha}_n(p)$ for any p and n. Fix an n and assume inductively that if $n < N$, $\alpha_{n+1}(p) = \tilde{\alpha}_{n+1}(p)$ for all p on-path.

Let V, $\tilde{V}$ denote the value functions under the equilibria associated with λ and $\tilde{\lambda}$, respectively. Note that $V_{p,n+1} = \tilde{V}_{p,n+1}$ for all p on-path. In the case where $n = N$, $V_{p,n+1} = \tilde{V}_{p,n+1} = 0$, and thus this holds trivially. In the case where $n < N$, this follows from the inductive assumption.

Now, note that by Lemma 5, α_n and $\tilde{\alpha}_n$ must both be a solution to (P) at all (p, n) on-path, which does not depend on λ. By Theorem 1, the solution to (P) is unique, and thus $\alpha_n(p) = \tilde{\alpha}_n(p)$ at all (p, n) on-path.

Now fixing any p and n, let $p(t)$ and $\tilde{p}(t)$ denote the common beliefs under λ and $\tilde{\lambda}$, respectively. It follows from (**??**) that $p(t) > \tilde{p}(t)$ for all $t > 0$. Thus, because $\alpha_n(p)$ and $\tilde{\alpha}_n(p)$ are both weakly decreasing in p (Proposition 3), it follows that $\alpha_n(p(t)) \leq \tilde{\alpha}_n(p(t))$. Furthermore, since $\tilde{\alpha}(p)$ is strictly decreasing in p (Proposition 3) whenever $\alpha_n(p) < 1$ and $k_N > \beta$, it follows that $\alpha_n(p(t)) < \alpha_n(\tilde{p(t)})$ in this case.

Finally, let us establish part (c). Let α and $\tilde{\alpha}$ denote the equilibria under N and $N+1$ firms, respectively, fixing all other parameters. We begin by showing that for all p, $\alpha_n(p) \geq \tilde{\alpha}_n(p)$, and $\alpha_n(p) > \tilde{\alpha}_n(p)$ when $\alpha_n(p) < 1$. To this end, fix an $n \in \{1, ..., N\}$ and assume inductively that the claim holds for $n+1$ whenever $n < N$.

We begin by showing that $\tilde{\alpha}_n(p) = 1$ implies that $\alpha_n(p) = 1$. Suppose that $\tilde{\alpha}_n(p) = 1$. By Proposition 1, $\beta > k_n$ and $p < \tilde{p}^*_n \equiv \frac{\beta - k_n}{\beta - k_n/(N+1-n)}$. Because $p^*_n \equiv \frac{\beta - k_n}{\beta - k_n/(N-n)} > \tilde{p}^*_n$, it follows from Proposition 1 that $\alpha_n(p) = 1$.

Now consider the case where $\tilde{\alpha}_n(p) < 1$. We wish tot show that $\tilde{\alpha}_n(p) < \alpha_n(p)$. To this end, we begin by making the following observation:

$$\text{If } \alpha_n \text{ and } \tilde{\alpha}_n \text{ both satisfy (ODE) at } q, \text{ and } \alpha_n(q) = \tilde{\alpha}_n(q), \text{ then } \alpha'_n(q) > \tilde{\alpha}'_n(q). \tag{30}$$

Let us now establish this. Note first that for α_n and $\tilde{\alpha}_n$ to both satisfy (ODE) at q, given that $\alpha_n(q) = \tilde{\alpha}_n(q)$, the following must hold:

$$\alpha'_n(q) = \frac{-1}{k_n(1-q)\alpha_n(q)} \frac{N-n}{N-n+1}(k_n\alpha_n(q) - V_{q^i,n+1} - \beta(1-\alpha_n(q))(1-q))$$

$$\tilde{\alpha}'_n(q) = \frac{-1}{k_n(1-q)\alpha_n(q)} \frac{N-n+1}{N-n+2}(k_n\alpha_n(q) - \tilde{V}_{q^i,n+1} - \beta(1-\alpha_n(q))(1-q)),$$

where V and $\tilde{V}$ denote the value functions under the equilibria with N and $N+1$ total firms, respectively. Note that if $n = N$, $\alpha'_n(q) = 0$. Meanwhile, by Proposition 3, $\tilde{\alpha}'_n(q) < 0$. Thus, $\tilde{\alpha}'_n(q) < \alpha_n(q)$ must hold. Next, consider the case where $n < N$. We begin by observting that $V_{q^i,n+1} > \tilde{V}_{q^i,n+1}$. To see why this must hold, first consider the case where $\tilde{\alpha}_{n+1}(q^i) = 1$. It then follows from the inductive assumption that $\alpha_n(q^i) = 1$. Then, by Lemma 5,

$$\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_\infty) = \frac{k_{n+1}q^i}{N-n} < \frac{k_{n+1}q^i}{N-n-1} = V_{q^i,n+1}(\delta_\infty) = V_{q^i,n+1}.$$

Next, consider the case where $\tilde{\alpha}_n(q^i) < 1$. In this case, it follows from Lemma 2 that

$$\begin{aligned}\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_0) = k_{n+1}\tilde{\alpha}_{n+1}(q^i) - \beta(1-q^i) < k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i) \\ = V_{q^i,n+1}(\delta_0) \le V_{q^i,n+1}\end{aligned}$$

where the strict inequality follows from the inductive assumption made above. Examining the two ODEs listed above, since by Proposition 3, $\alpha_n'(q) \le 0$, it follows that $\tilde{\alpha}_n'(q) < \alpha_n'(q)$.

Now, assume by contradiction that $\alpha_n(p) \le \tilde{\alpha}_n(p)$. We begin by showing that there exists a $q^* < p$ such that $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$. First consider the case where $k_n \ge \beta$. Then, by Proposition 2,

$$\lim_{q\to 0+} \alpha_n(q) = \lim_{q\to 0+} \tilde{\alpha}_n(q) = \frac{\beta}{k_n}$$

Then, by the continuous differentiability of α_n and $\tilde{\alpha}_n$ on $(0,p)$, it follows from Equation 30 that for some $q^* < p$ sufficiently small $\alpha_n(q^*) > \tilde{\alpha}_n(q^*)$. Next, consider the case where $k_n < \beta$, and let $p_n^* \equiv \frac{\beta-k_n}{\beta/(N-n+1)-k_n}$. Note by Proposition 1 that $\alpha_n(p_n^*) = 1$. Meanwhile, because $p_n^* < \tilde{p}_n^* \equiv \frac{\beta-k_n}{\beta/(N-n+2)-k_n}$, it follows from Proposition 1 that $\tilde{\alpha}_n(p_n^*) < 1$, and thus, we have for $q^* = p_n^*$, $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$.

Since $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$ and $\tilde{\alpha}_n(p) \ge \alpha_n(p)$, by the continuous differentiability of α on $[q^*,p]$, there must exist some $q \in (q^*,p]$ such that $\alpha_n(q) = \tilde{\alpha}_n(q)$ and $\alpha_n'(q) \le \tilde{\alpha}_n'(q)$. However, this is a contradiction of (30).

Now fixing any p and n, let $p(t)$ and $\tilde{p}(t)$ denote the common beliefs under N and $N+1$ firms, respectively. We wish to show that on some interval $[0,\bar{t}]$, where $\bar{t} > 0$, $\alpha_n(p(t)) \ge \tilde{\alpha}_n(\tilde{p}(t))$ is weakly increasing in t, and strictly so whenever $\alpha_n(p(t)) < 1$. First consider the case where $\alpha_n(p(t)) = 1$. In this case, the statement holds trivially. Next, consider the case where $\alpha_n(p) < 1$. It follows from the above that $\alpha_n(p) > \tilde{\alpha}_n(p)$. Now note that it follows from (**??**) that $\lim_{t\to 0+} p(t) - \tilde{p}(t) = 0$. Since $\alpha_n(p(t))$ and $\tilde{\alpha}_n(\tilde{p}(t))$ are both continuous in t (Lemma 4), it follows that for some $\bar{t} > 0$, $\alpha_n(p(t)) > \tilde{\alpha}_n(\tilde{p}(t))$ for all $t \in [0,\bar{t}]$. □

Appendix F Proofs: heterogenous learning abilities

Here, we consider the extended model presented in Section 7. The objective is to establish Proposition 6. This proof will require extending certain results established in the baseline model to the extended model.

Regarding Lemmas 1-4, I will take for granted that these hold under the extended model. Formal proofs of this are omitted as all proofs presented under the baseline model will apply to the extended setting as well.

Next, I establish that Proposition 1 holds under the extended model. This claim is presented as Proposition 1′. In the analysis below, I let $V^i_{p,n}$ denote firm i's value.

Proposition 1′. *For all s, there exists a $p^{i*} \in (0, 1]$ such that at any p on-path, $\alpha^i_1(p) = 1$ if and only if the following two conditions hold:*

1. $k_1 \leq \beta$
2. $p \leq p^{i*}$

Furthermore, $p^{j*} > p^{i*}$ whenever $\lambda^j > \lambda^i$ and $n > 1$.

Proof. Fix an i. Suppose that $k_1 \leq \beta$. By identical reasoning as Proposition 1, for all $q < \frac{\beta - k_1}{k_1}$, $\alpha^i_1(q) = 1$. Let

$$p^{i*} \equiv sup\{p|\alpha^i_1(p) = 1 \text{ for all } q < p\}$$

It follows by definition that $\alpha^i_1(p) = 1$ for all $p \leq p^{i*}_1$.

Next, we will show that $\alpha^i_1(q) < 1$ whenever $k_1 > \beta$ or $p > p^{i*}_1$. Suppose not by contradiction. First, consider the case where $k_1 > \beta$ and $\alpha^i_1(p) = 1$ for some p. Then we have that

$$V^i_{p,1}(\delta_0) = k_1 p + (k_1 - \beta)(1 - p) > k_1 p \leq V^i_{p,1}(\delta_\infty)$$

Thus, i can profitably deviate at p. Contradiction. Next, consider the case where $q > p^{i*}_n$ and $\alpha^i_1(p) = 1$. In this case, a contradiction follows from identical reasoning to what is presented in Proposition 1.

Finally, we show that $p^{j*} > p^{i*}$ whenever $\lambda^j > \lambda^i$. Suppose by contradiction that $p^{j*} \leq p^{i*}$. Note that because j is truth telling at $(p^{j*}_S, n = 1)$, $V^j_{p^{j*}_1,1}(\delta_\infty) \geq V^j_{p^{j*},1}(\delta_0)$. Furthermore, because $p^{j*} \leq p^{i*}$, i is also truthful at $(p^{j^*}_n, n = 1)$. Thus,

$$V^j_{p^{j*}_1,1}(\delta_0) = V^i_{p^{j*}_1,1}(\delta_\infty) = k_1 - \beta(1 - p).$$

Now, note that because $\lambda^j > \lambda^i$,

$$V^j_{p^{j*}_1,1}(\delta_\infty) > V^i_{p^{j*}_1,1}(\delta_\infty).$$

Combining these inequalities we have $V^i_{p^{j*}_1,1}(\delta_\infty) < V^i_{p^{j*}_1,1}(\delta_0)$. However, because $\alpha^i_1(p^{j*}) = 1$, $V^j_{p^{j*}_n,1} = V^j_{p^{j*}_n,1}(\delta_\infty)$. Contradiction. $\square$

Next, we extend Proposition 2 to this setting. Note this entails deriving an ODE that applies to this extended model, (ODE′).

Proposition 2′. *In equilibrium, for any p on-path, if $k_1 \geq \beta$ or $p > p^{i*}$, then the following must be satisfied:*

$$\alpha_1^{i\prime}(p) = -\beta - \frac{\sum_{j \neq i} \frac{\lambda^j}{\alpha_1^j(p)}}{\sum_j \lambda^j (1-p)} [\alpha^i(p) - \beta(1-p)] \tag{ODE′}$$

In addition, $\lim_{p \to 0+} \alpha_1^i(p) = \beta / k_1$ must hold if $k_1 > \beta$, and $\lim_{p \to p^{i*}+} \alpha_1^i(p) = 1$ if $k_1 \leq \beta$.

Proof. Let us first establish that (ODE′) must hold under the conditions specified.

When $k_1 \geq \beta$ or $p > p^{i*}$, it follows from Proposition 1′ that $\alpha_1^i(p(t)) < 1$. It then follows from Lemma 2 that there exsits an $\varepsilon > 0$ such that for all $\Delta \in (0, \varepsilon)$,

$$\frac{V_{p,1}^i(\delta_\Delta) - V_{p,1}^i(\delta_0)}{\Delta} = 0$$

Recall that $V_{p,1}^i(\delta_0) = k_1 \alpha_1^i(p) - \beta(1-p)$. Meanwhile,

$$V_{p,1}^i(\delta_\Delta) = \int_0^\Delta k_1 \alpha_1^i(p(s)) \Psi^i(s) ds + (1 - \sum_j \lim_{s \to \Delta-} \Psi^j(s))[k_1 \alpha_1(p(\Delta)) - \beta(1 - p(\Delta))]$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p,1}$. Specifically, for all $s > 0$,

$$\Psi^i(s) = p\lambda^i \int_0^s e^{-\prod_{j \in S} \lambda^j r} \prod_{j \neq i} (1 - F_{p,1}^i(r)) dr$$

and for $j \neq i$,

$$\Psi^j(s) = p \int_0^s e^{-\prod_{k \neq j} \lambda^k r} \prod_{k \neq i \neq j} (1 - F_{p,1}^k(r)) d(-e^{-\lambda^j r}(1 - F_{p,1}^j(r)))$$
$$+ (1-p) \int_0^s \prod_{k \neq i \neq j} (1 - F_{p,1}^k(r)) dF_{p,1}^j(r)$$

Substituting these two expressions into the above equation for $V_{p,1}^i(\delta_0)$ and following the same sequence of steps in Proposition 2 yields (ODE′).

The two limit conditions are established by the same reasoning presented in Proposition 2. □

Proof of Proposition 6. Fix any (i, j) such that $\lambda^i > \lambda^j$. We want to show that $\alpha_1^i(p(t)) \leq \alpha_i^j(p(t))$ and that $\alpha_1^i(p(t)) < \alpha_i^j(p(t))$ whenever $\alpha_1^i(p(t)) < 1$. First suppose $\alpha_1^i(p) = 1$. In this case, $\alpha_1^i(p) \geq \alpha_1^j(p)$ is trivially satisfied.

Next, suppose $\alpha_1^i(p) < 1$. We want to show that $\alpha_1^i(p) > \alpha_1^j(p)$. Suppose by contradiction that $\alpha_1^i(p) \leq \alpha_1^j(p)$. First consider the case where $k_1 < \beta$. Then, let

$$q^* \equiv \inf\{q|\alpha_1^j(p) < 1 \text{ and } \alpha_1^j(p) < \alpha_1^i(p)\}.$$

Because the α_1^i are continuous, it follows from Proposition 1′, and the assumption that $\alpha_1^i(p) \leq \alpha_1^j(p)$, that $q^* < p$ exists. Again, by continuity, $\alpha_1^j(q^*) = \alpha_1^i(q^*)$. It then follows from (ODE′) that $\alpha_1^{j\prime}(q^*) < \alpha_1^{i\prime}(q^*)$. But this implies that for some $q > q^*$, $\alpha_1^j(q^*) > \alpha_1^i(q^*)$. Contradiction.

Next, consider the case where $k_1 \geq \beta$. Recall by Proposition 2′ that $\lim_{p\to 0+} \alpha_1^i(p) = \lim_{p\to 0+} \alpha_1^j(p)$. Thus, there exists some $q \in (0, p]$ such that $\alpha_1^i(p) \leq \alpha_1^j(p)$ and $\alpha_1^{i\prime}(p) \leq \alpha_1^{j\prime}(p)$. However, it again follows from (ODE′) that $\alpha_1^{i\prime}(p) > \alpha_1^{j\prime}(p)$. Contradiction.

□