Supplemental Appendix

Proofs of Lemmas

Proof of Lemma 1. Let us begin by showing that at all (p, n) on-path such that $p < 1$, $F_{p,n}$ is continuous at 0. To this end, suppose by contradiction that $F_{p,n}$ is discontinuous at 0. By the right-continuity of $F_{p,n}$, this implies that $F_{p,n}(0) > 0$. Because (p, n) is on path, by (4), $\alpha_n(p) = 0$. Furthermore, it follows by definition that $p^i(0) = p$. Recalling that we are restricting attention to symmetric equilibria, let Ψ denote the first-report distribution at (p, n) under the equilibrium strategy profile $F_{p,n}$. Because $F_{p,n}(0) > 0$, $\Psi^i(0) > 0$ for all i who have not yet reported.

Now define the following deviation $\hat{F}_{p,n}$. This strategy is identical to $F_{p,n}$, except that all the mass that $F_{p,n}$ places on 0 is shifted to ∞:

$$\hat{F}_{p,n}(s) = \begin{cases} F_{p,n}(s) - F_{p,n}(0) & \text{if } s < \infty \\ 1 & \text{if } s = \infty. \end{cases} \tag{20}$$

Now, fix some i who has not yet reported. Let $\hat{\Psi}$ denote the first-report distribution at (p, n) under the strategy profile where i plays $\hat{F}_{p,n}$ and all $j \neq i$ play $F_{p,n}$. By definition, for all $s \geq 0$, $\hat{\Psi}^i(s) = \Psi^i(s) - \Psi^i(0)$. Then,

$$\int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\hat{\Psi}^i(s) > \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s).$$

Again by definition, for all $s \geq 0$, $\hat{\Psi}^{-i}(s) = \Psi^{-i}(s) + X(s)$, where

$$X(s) \equiv \Psi^i(0)[p \int_0^s (1 - F_{p,n})^{N-n-1}(1 - \hat{F}_{p,n}(r)) e^{-\lambda r(N-n)} d(e^{-\lambda r}(F_{p,n}(r) - 1))$$
$$+ (1 - p) \int_0^s (1 - F_{p,n}(r))^{N-n-1}(1 - \hat{F}_{p,n}(r)) dF_{p,n}(r)].$$

Then, we have

$$\int_0^\infty V_{p^{-i}(s),n+1} d\hat{\Psi}^{-i}(s) - \int_0^\infty V_{p^{-i}(s),n+1} d\Psi^{-i}(s) = \int_0^\infty V_{p^{-i}(s),n+1} dX(s) \geq 0.$$

where the final inequality follows from $X(s)$ increasing in s and $V_{p^{-i}(s),n+1} \geq$

$V_{p^{-i}(s),n+1}(\delta_\infty) \geq 0$. Combining the above two inequalities we have

$$V_{p,n}(\hat{F}_{p,n}) = \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\hat{\Psi}^i(s) + (N-n)\int_0^\infty V_{p^{-i}(s),n+1}d\hat{\Psi}^{-i}(s)$$
$$> \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\Psi^i(s) + (N-n)\int_0^\infty V_{p^{-i}(s),n+1}d\Psi^{-i}(s) = V_{p,n}(F_{p,n}).$$

Thus, i can profitably deviate at (p,n). Contradiction.

It remains to show that continuity applies at all t, for all (p,n) on-path such that $p<1$. Suppose by contradiction that it is not. Let t denote the time at which there is a discontinuity. Because $F_{p,n}$ is increasing and right-differentiable $\lim_{r\to t-} F_{p,n}(r) < F_{p,n}(t)$. By (3), $F_{p(t),n}(0) > 0$. Thus, $F_{p(t),n}$ is discontinuous at 0. Contradiction. □

I now state and prove a technical lemma (Lemma 6),which is used in the proof of Lemma 2.

Lemma 6. *For any (p,n) on-path, $\alpha_n(p) \geq \underline{\alpha}_n(p) \equiv \min\{\beta(1-p)/k_n, 1\}$ and $F'_{p,n}(0+) \leq \overline{f} \equiv \lambda p(\frac{1}{\underline{\alpha}_n(p)} - 1)$.*

Proof of Lemma 6. We begin by showing the first point above. The second point follows by definition of $\alpha_n(p)$.

First, suppose by contradiction that there exists a (p,n) on-path such that $\alpha_n(p) < \min\{\beta(1-p)/k_n, 1\}$. Recalling that $p(s)$ is given by (2), I begin by showing that for all s sufficiently small, $(p(s),n)$ is on-path. Suppose not by contradiction. Since (p,n) is on-path, this implies that $F_{p,n}(s) = 1$, which contradicts Lemma 1. It thus follows from (4), combined with the piecewise twice differentiability and right-differentiability of $F_{p,n}$, that $\alpha_n(p(s))$ is continuous in some right-neighborhood of $s=0$. Thus, there exists an $\varepsilon > 0$ such that for all $s \in [0,\varepsilon]$, $k_n\alpha_n(p(s)) < \beta(1-p)$.

Next, I claim that $F_{p,n}(\varepsilon) > 0$. Suppose this is not true by contradiction. Then, it follows that $F_{p,n}(s) = 0$ for all $s \in [0,\varepsilon]$, implying by definition of α that $\alpha_n(p) = 1$, contradicting the assumption that $\alpha_n(p) < 1$. Now, define the following deviation $\tilde{F}_{p,n}$, which shifts the mass $F_{p,n}$ places on $[0,\varepsilon]$ to ∞:

$$\tilde{F}_{p,n}(s) = \begin{cases} 0 & \text{if } s \in [0,\varepsilon] \\ F_{p,n}(s) - F_{p,n}(\varepsilon) & \text{if } s \in (\varepsilon,\infty) \\ 1 & \text{if } s = \infty. \end{cases}$$

The admissibility (i.e., right-continuity and piecewise twice-differentiability) of $\tilde{F}_{p,n}$ follows from the admissibility of $F_{p,n}$. We now wish to show that $\tilde{F}_{p,n}$ is a profitable deviation at (p, n). Let Ψ denote the first-report distribution under the strategy profile where all players play $F_{p,n}$, and let $\tilde{\Psi}$ denote the first-report distribution under the strategy profile where i plays $\tilde{F}_{p,n}$ and all $j \neq i$ play $F_{p,n}$.

By definition of Ψ, $\tilde{\Psi}^i(s) = \Psi^i(s) - X(s)$, where

$$X(s) = \begin{cases} p \int_0^s e^{-\lambda r(N-n)}(1 - F_{p,n}(r))^{N-n} d(e^{-\lambda r}(F_{p,n}(r) - 1)) \\ +(1-p) \int_0^s (1 - F_{p,n}(r))^{N-n} dF_{p,n}(r) & \text{if } s \in [0, \varepsilon] \\ X(\varepsilon) & \text{if } s > \varepsilon. \end{cases}$$

$X(s)$ is weakly increasing in s. Furthermore, because $F_{p,n}(\varepsilon) > 0$, it follows that $F_{p,n}(s)$ strictly increases on $[0, \varepsilon]$. Thus, $X(s)$ is strictly increasing at some $s \in [0, \varepsilon]$. Now, by the above definition:

$$\int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\tilde{\Psi}^i(s) - \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s)$$
$$= - \int_0^\varepsilon [k_n \alpha_n(p(s)) - \beta(1 - p(s))] dX(s) > 0.$$

where the strict inequality follows from the fact that $X(s)$ is strictly increasing on $[0, \varepsilon]$ and the above-established fact that $k_n \alpha_n(p(s)) < \beta(1 - p(s))$ for all $s \in [0, \varepsilon]$.

Next, let us consider $\tilde{\Psi}^{-i}(s)$. By definition, $\tilde{\Psi}^{-i}(s) = \Psi^{-i}(s) - Y(s)$, where

$$Y(s) = -p \int_0^s [e^{-\lambda r}(1 - F_{p,n}(r))]^{n-2} F_{p,n}(\min\{r, \varepsilon\}) d(e^{-\lambda r}(F_{p,n}(r) - 1)) -$$
$$(1-p) \int_0^s (1 - F_{p,n}(r))^{n-2} F_{p,n}(\min\{r, \varepsilon\}) dF_{p,n}(r).$$

Thus,

$$\int_0^\infty V_{p^{-i}(s),n+1} d\tilde{\Psi}^{-i}(s) - \int_0^\infty V_{p^{-i}(s),n+1} d\Psi^{-i}(s) = \int_0^\infty V_{p^{-i}(s),n+1} dY(s) \geq 0.$$

where the final inequality follows from the fact that $Y(s)$ is weakly increasing in s and $V_{p^{-i}(s),n+1} \geq 0$. Combining the previous two inequalities, we obtain that $V_{p,n}(\tilde{F}_{p,n}) > V_{p,n}(F_{p,n})$, and thus i can profitably deviate at (p, n). Contradiction. □

Proof of Lemma 2. Assume that $\alpha_n(p) < 1$. By the right continuity and piecewise twice-differentiability of $F_{p,n}$, and by (4), it follows that $\alpha_n(p(s))$ is right-continuous in s. Thus, there exists an $\varepsilon > 0$ and $d > 0$ such that $\alpha_n(p(s)) < 1-d$ for all $s \in [0, \varepsilon)$. I claim that for all $s \in [0, \varepsilon)$, $V_{p,n} = V_{p,n}(\delta_s)$. Suppose by contradiction that for some $\tilde{s} \in [0, \varepsilon)$,$V_{p,n}(\delta_{\tilde{s}}) < V_{p,n}$. I show that $V_{p,n}(\delta_s)$ is right-continuous in s. By definition,

$$V_{p,n}(\delta_s) = \int_0^s k_n \alpha_n(p(r)) d\Psi^i(r) + (N-n) \int_0^s V_{p^i(r),n} d\Psi^{-i}(r) + \\ (1 - \sum_j \Psi^j(s))[k_n \alpha_n(p(s)) - \beta(1-p(s))],$$

where $\Psi^j(s)$ is the first-report distribution that arises when i plays δ_∞ and all $j \neq i$ play $F_{p,n}$. The right-continuity of $V_{p,n}(\delta_s)$ with respect to s then follows from the absolute continuity of Ψ^j (which follows from Lemma 1), and the right-continuity of $\alpha_n(p(s))$ with respect to s, which follows from the right-continuity of $F_{p,n}(s)$.

Given the right continuity of $V_{p,n}(\delta_s)$, there exists some $\varepsilon' \in (0, \varepsilon - \tilde{s})$ and $x > 0$ such that $V_{p,n} - V_{p,n}(\delta_r) > x$ for all $r \in [\tilde{s}, \tilde{s} + \varepsilon']$. I claim there exists some $s^* \in [0, \infty]$ such that $V_{p,n} = V_{p,n}(\delta_{s^*})$. Suppose by contradiction that $V_{p,n} > V_{p,n}(\delta_s)$ for all $s \in [0, \infty]$. It follows from (5) that

$$V_{p,n}(F) = \int_0^\infty V_{p,n}(\delta_s) dF_{p,n}(s) + (1 - \lim_{s \to \infty} F_{p,n}) V_{p,n}(\delta_\infty) < V_{p,n},$$

where the strict inequality follows from the assumption that $V_{p,n} > V_{p,n}(\delta_s)$ for all s. Thus, F cannot be an equilibrium strategy. Contradiction.

Now, define the following deviation $\tilde{F}$. This strategy is identical to F, except $\tilde{F}_{p,n}$ shifts all the mass from $[s, s+\varepsilon']$ to s^*. Specifically, when $s^* < \tilde{s}$:

$$\tilde{F}_{p,n}(t) = \begin{cases} F_{p,n}(t) + F_{p,n}(\tilde{s} + \varepsilon) - F_{p,n}(\tilde{s}) & \text{if } t \in [s^*, \tilde{s}] \\ F_{p,n}(\tilde{s} + \varepsilon) & \text{if } t \in (\tilde{s}, \tilde{s} + \varepsilon'] \\ F_{p,n}(t) & \text{otherwise.} \end{cases}$$

Meanwhile, when $s^* > \tilde{s} + \varepsilon'$:

$$\tilde{F}_{p,n}(t) = \begin{cases} F_{p,n}(s) & \text{if } t \in [\tilde{s}, \tilde{s} + \varepsilon] \\ F_{p,n}(t) - [F_{p,n}(\tilde{s} + \varepsilon') - F_{p,n}(\tilde{s})] & \text{if } t \in (\tilde{s} + \varepsilon', s^*) \\ F_{p,n}(t) & \text{otherwise.} \end{cases}$$

Now, by definition:

$$V_{p,n}(\tilde{F}) = V_{p,n}(F) + \int_{\tilde{s}}^{\tilde{s}+\varepsilon'} [V_{p,n}(\delta_{s^*}) - V_{p,n}](\delta_r) dF_{p,n}(r) \geq V_{p,n}(F) + x\varepsilon' > V_{p,n}(F_{p,n}).$$

Thus, $\tilde{F}$ is a profitable deviation. Contradiction.

It remains to show that $V_{p,n} = V_{p,n}(\delta_\infty)$. Suppose by contradiction that $V_{p,n} > V_{p,n}(\delta_\infty)$. It follows that $\lim_{t\to\infty} F_{p,n}(t) = 0$, because otherwise, the firm could profitably deviate by placing no mass on $t = \infty$. Thus, for some $s \in (0,\infty]$, $\lim_{t\to s-} b_n(p(t)) = \infty \Rightarrow \lim_{t\to s-} \alpha_n(p(t)) = 0$, which contradicts Lemma 6. □

Proof of Lemma 3. Fix a (p,n) on-path. I first show that for all $s \geq 0$,

$$\alpha_n(p(s)) = \frac{\lambda p(s)}{\lambda p(s) + \frac{F'_{p,n}(s+)}{1-F_{p,n}(s)}} \tag{21}$$

It follows from Lemma 6 that $(p(s), n)$ is on-path for all $s \geq 0$. Thus, by Lemma 1, $F_{p(s),n}(0) = 0$, and by (4) $\alpha_n(p(s)) = \frac{\lambda p(s)}{\lambda p(s) + F'_{p(s),n}(0+)}$. Next, it follows from (3) that $F'_{p(s),n}(0+) = \frac{F'_{p,n}(s+)}{1-F_{p,n}(s)}$. Combining the previous two equations yields (21). It thus follows from the right-differentiability and piecewise twice-differentiability of $F_{p,n}$ that $\alpha_n(p(s))$ is right-continuous in s. It remains to show that $\alpha_n(p(s))$ is left-continuous in s. Suppose by contradiction there exists an s such that $\alpha_n(p(s))$ is left-discontinuous. Then there exists some $d > 0$ such that for all $\varepsilon > 0$, there exists an $s_\varepsilon \in (s-\varepsilon, s)$ such that $|\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s))| > d$. First consider the case where for all $\varepsilon > 0$, there exists an $s_\varepsilon \in (s-\varepsilon, s)$ such that $\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s)) > d$. I begin by claiming that for all $\varepsilon > 0$,

$$V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}). \tag{22}$$

Note that there exists some $s^* \in (s,\infty]$ such that $V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_{s^*-s_\varepsilon})$. To see why this must hold, suppose not, by contradiction. Then it must be that $F_{p(s_\varepsilon),n}$ places full mass on $[0, s-s_\varepsilon]$, and thus, either Lemma 1 or (6) would be violated. Thus, we have

$$
\begin{aligned}
V_{p(s_\varepsilon),n} &= \int_0^{s-s_\varepsilon} k_n\alpha_n(p(s_\varepsilon+r))d\Psi^i(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(s_\varepsilon+r),n+1}d\Psi^{-i}(r) \\
&+ (1-\sum_j \Psi^j(s-s_\varepsilon))V_{p(s),n}(\delta_{s^*-s}) \\
&= \int_0^{s-s_\varepsilon} k_n\alpha_n(p(s_\varepsilon+r))d\Psi^i(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(s_\varepsilon+r),n+1}d\Psi^{-i}(r) \\
&+ (1-\sum_j \Psi^j(s-s_\varepsilon))V_{p(s),n}(\delta_0) = V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}),
\end{aligned}
$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p(s_\varepsilon),n}$. Note that the equality follows from the fact that $\alpha_n(p(s)) < 1$, and thus by Lemma 2, $V_{p(s),n} = V_{p(s),n}(\delta_0)$. However, note that for all $\varepsilon > 0$,

$$
\begin{aligned}
V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = \int_0^{s-s_\varepsilon} k_n\alpha_n(p(s_\varepsilon+r))d\Psi^i(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(s_\varepsilon+r),n+1}d\Psi^{-i}(r) \\
+(1-\sum_j \Psi^j(s-s_\varepsilon))[k_n\alpha_n(p(s),n) - \beta(1-p(s))].
\end{aligned}
$$

Because the Ψ^j are absolutely continuous,

$$
\lim_{\varepsilon\to 0} V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = k_n\alpha_n(p(s),n) - \beta(1-p(s)).
$$

Then, by the assumption that $\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s)) > d$, for all $\varepsilon > 0$ sufficiently small $V_{p(s_\varepsilon),n}(\delta_0) = k_n\alpha_n(p(s_\varepsilon),n) - \beta(1-p(s_\varepsilon)) > V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon})$, contradicting (22).

Next, consider the case where for all $\varepsilon > 0$, $\alpha_n(p(s)) - \alpha_n(p(s_\varepsilon)) > d$. As shown above, $\lim_{\varepsilon\to 0} V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = V_{p(s),n}(\delta_0)$. Thus, for ε sufficiently small,

$$
V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) > k_n\alpha_n(p(s_\varepsilon)) - \beta(1-p(s_\varepsilon)) = V_{p(s_\varepsilon),n}(\delta_0).
$$

However, since $\alpha_n(p(s_\varepsilon)) < 1$ for all $\varepsilon > 0$, by Lemma 2, $V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_0)$. Contradiction. □

Proof of Lemma 4. Fix an (α, F). I begin by establishing the necessity of the three conditions specified in Definition 2 for (α, F) to be an equilibrium. First we establish the necessity of part 3. of Definition 2. To this end, recall that by the selection assumption, $F_{1,n}(0) = 1$. Thus, it follows from (4) that $\alpha_n(1) = 0$ if $(p = 1, n)$ is on-path. Parts 1. and 2. of Definition (2) follow immediately from Proposition 1 and Proposition 2, respectively.

Next, we establish the sufficiency of the above conditions for (α, F) to be an equilibrium. We begin by considering the case in which $k_n < \beta$ and $p \leq p_n^*$. It follows from Definition 2 that $\alpha_n(q) = 1$ for all $q \leq p$. Thus, by (4), $F_{p,n} = \delta_\infty$. We want to show that there exist no profitable deviations in this case, i.e., that $V_{p,n} = V_{p,n}(\delta_\infty)$. It suffices to show that

$$V_{p,n}(\delta_\infty) \geq V_{p,n}(\delta_s) \text{ for all } s \in [0, \infty). \tag{23}$$

First, note that for all $s \in (0, \infty)$,

$$V_{p,n}(\delta_s) = k_n(1-p(1-e^{-\lambda s(N-n+1)})(\frac{N-n}{N-n+1})) - \beta(1-p) \leq k_n - \beta(1-p) = V_{p,n}(\delta_0).$$

Further, $k_n \leq \beta$ and $p \leq p_n^*$ implies that $V_{p,n}(\delta_0) = k_n - \beta(1-p) \leq \frac{k_n}{N-n+1} = V_{p,n}(\delta_\infty)$. Thus, $V_{p,n}(\delta_\infty) \geq V_{p,n}(\delta_s)$ for all $s \in [0, \infty)$.

Next, we show that $F_{p,n}$ is optimal when $k_n > \beta$ or $p \geq p_n^*$. We begin by showing

$$\frac{d}{d\Delta} V_{p,n}(\delta_\Delta) = 0 \text{ for all } \Delta \in [0, \infty) \text{ if } k_n \geq \beta \text{ and for all } \Delta \in [0, t^*) \text{ if } k_n < \beta \tag{24}$$

where t^* is the unique solution to $p(t^*) = p_n^*$. Note that

$$\begin{aligned} V_{p,n}(\delta_\Delta) = \int_0^\Delta k_n \alpha_n(p(s)) d\Psi^i(s) + (N-n) \int_0^\Delta V_{p^i(s),n+1} d\Psi^{-i}(s) + \\ (1 - \sum_j \Psi^j(\Delta))(k_n \alpha_n(p(\Delta)) - \beta(1 - p(\Delta))), \end{aligned} \tag{25}$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p,n}$. Then,

$$\begin{aligned} \frac{d}{d\Delta} V_{p,n}(\delta_\Delta) &= (N-n)[V_{p^i(\Delta),n+1} - k_n \alpha_n(p(\Delta)) + \beta(1-p(\Delta))] \Psi^{-i\prime}(\Delta) - \beta(1-p(\Delta)) \Psi^{i\prime}(\Delta) \\ &+ (1 - \sum_j \Psi^j(\Delta)) p'(\Delta)(k_n \alpha_n'(p(\Delta)) + \beta), \end{aligned}$$

where $\Psi^{i\prime}(t) \equiv \frac{d}{dt} \Psi^i(t)$.

In the above, the existence of $\Psi^{j\prime}(\Delta)$ follows from the differentiability of α_n at $p(\Delta)$, and thus, the differentiability of $F_{p,n}$ at Δ. We wish to show that $\frac{d}{d\Delta}V_{p,n}(\delta_\Delta) = 0$. To this end, we begin by deriving expressions for $\Psi^{i\prime}(\Delta)$ and $\Psi^{-i\prime}(\Delta)$. First, it follows by definition of the first-report distribution that:

$$\Psi^i(\Delta) = p\lambda \int_0^\Delta (1 - F_{p,n}(s))^{N-n} e^{-\lambda(N-n+1)s} ds.$$

Differentiating this, we obtain:

$$\Psi^{i\prime}(\Delta) = p\lambda(1 - F_{p,n}(\Delta))^{N-n} e^{-\lambda(N-n+1)\Delta}.$$

Meanwhile:

$$\Psi^{-i}(\Delta) = p \int_0^\Delta (1-F_{p,n}(s))^{N-n-1} e^{-\lambda(N-n)s} d((F_{p,n}(s)-1)e^{-\lambda s}) + (1-p) \int_0^\Delta (1-F_{p,n}(s))^{N-n-1} F'_{p,n}(s) ds.$$

where the existence of $F'_{p,n}(s)$ again follows from the assumption that α_n is differentiable at $p(s)$. Differentiating this, we obtain:

$$\Psi^{-i\prime}(\Delta) = (1 - F_{p,n}(\Delta))^{N-n} [\frac{F'_{p,n}(\Delta)}{1 - F_{p,n}(\Delta)} (pe^{-\lambda\Delta(N-n+1)} + (1-p)) + pe^{-\lambda\Delta(N-n+1)}\lambda].$$

It follows from (4) and (3) that

$$\frac{F'_{p,n}(\Delta)}{1 - F_{p,n}(\Delta)} = \lambda p(\Delta)(\frac{1}{\alpha_n(p(\Delta))} - 1).$$

Substituting this, along with the definition of $p(\Delta)$, we obtain:

$$\Psi^{-i\prime}(\Delta) = \lambda(1 - F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta(N-n+1)} + (1-p)) \frac{p(\Delta)}{\alpha_n(p(\Delta))}.$$

Note further that

$$1 - \sum_j \Psi^j(\Delta) = (1 - F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta(N-n+1)} + (1-p)). \tag{26}$$

Substituting the expressions for $\Psi^{i\prime}(\Delta)$, $\Psi^{-i\prime}(\Delta)$, and $1 - \sum_j \Psi^j(\Delta)$ into the expression for $\frac{d}{d\Delta}V_{p,n}(\delta_\Delta)$ we obtain:

$$\frac{d}{d\Delta}V_{p,n}(\delta_\Delta) = K[\frac{(N-n)}{\alpha_n(p(\Delta))}(V^i_{p(\Delta),n+1} - k_n\alpha_n(p(\Delta)) + \beta(1-p(\Delta))(1-\alpha_n(p(\Delta)))) \\ -k_n\alpha_n'(p(\Delta))(1-p(\Delta))(N-n+1)].$$

where $K \equiv \lambda(1-F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta(N-n+1)} + (1-p))p(\Delta)$. Because (ODE) is satisfied at $(p(\Delta), n)$, using it to substitute in for $\alpha_n'(p(\Delta)$, we obtain (24).

Now, consider the case where $k_n \geq \beta$. To show $F_{p,n}$ is optimal, it suffices to show that all pure strategies δ_Δ yield the same payoff, i.e., that

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_\Delta) \tag{27}$$

for all $\Delta \in [0,\infty]$. It follows from (24) that (27) holds for all $\Delta \in [0,\infty)$. It remains to show that (27) holds for $\Delta = \infty$. By (24),

$$\begin{aligned} V_{p,n}(\delta_0) &= \lim_{\Delta\to\infty} V_{p,n}(\delta_\Delta) \\ &= \lim_{\Delta\to\infty}\int_0^\Delta k_n\alpha_n(p(s))d\Psi^i(s) + (N-n)\lim_{\Delta\to\infty}\int_0^\Delta V_{p^i(s),n+1}d\Psi^{-i}(s) + \\ &\lim_{\Delta\to\infty}(1-\sum_j \Psi^j(\Delta))(k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta))) \\ &= \int_0^\infty k_n\alpha_n(p(\Delta))d\Psi^i(s) + (N-n)\int_0^\infty V_{p^i(\Delta),n+1}d\Psi^{-i}(s) = V_{p,n}(\delta_\infty), \end{aligned}$$

where the third equality follows from the limit condition $\lim_{p\to 0+}\alpha_n(p) = \beta/k_n$.

Finally, consider the case where $k_n < \beta$ and $p > p_n^*$. Because $\alpha_n(p(s)) = 1$ for all $s > t^*$, by (4), it follows that $F'_{p,n}(s) = 0$ for all $s > t^*$. Thus, the support of $F_{p,n}$ is a subset of $[0,t^*] \cup \infty$. Thus, to show $F_{p,n}$ is optimal, it suffices to show that δ_Δ is optimal for $\Delta \in [0,t^*] \cup \infty$. I first show that

$$V_{p,n}(\delta_\Delta) = V_{p,n}(\delta_0) \text{ for all } \Delta \in [0,t^*] \cup \infty \tag{28}$$

and then show

$$V_{p,n}(\delta_{t^*}) \geq V_{p,n}(\delta_\Delta)\text{for all } \Delta \in (t^*,\infty). \tag{29}$$

To show (28), recall that it follows from (24) that

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_\Delta) \text{ for all } \Delta \in [0, t^*).$$

It remains to show $V_{p,n}(\delta_0) = V_{p,n}(\delta_s)$ for $s \in \{t^*, \infty\}$. For $s = t^*$, it follows from the above that

$$V_{p,n}(\delta_0) = \lim_{\Delta \to t^*-} V_{p,n}(\delta_\Delta) = V_{p,n}(\delta_{t^*}),$$

where the final inequality follows from (25), and the continuity of $\alpha_n(p(t))$ and Ψ^j at t^*. I will now show $V_{p,n}(\delta_{t^*}) = V_{p,n}(\delta_\infty)$. Note that for all $\Delta \in [t^*, \infty]$:

$$V_{p,n}(\delta_\Delta) = \int_0^{t^*} k_n \alpha_n(p(s)) d\Psi^i(s) + (N-n) \int_0^{t^*} V_{p^i(s),n+1} d\Psi^{-i}(s) + (1 - \sum_j \Psi^j(t^*)) V_{p_n^*,n}(\delta_{\Delta - t^*}).$$

Thus, to show $V_{p,n}(\delta_{t^*}) = V_{p,n}(\delta_\infty)$, it suffices to show $V_{p_n^*,n}(\delta_0) = V_{p_n^*,n}(\delta_\infty)$. It follows from the definition of p_n^* that:

$$V_{p_n^*,n}(\delta_0) = k_n - \beta(1 - p_n^*) = \frac{k_n p_n^*}{n} = V_{p_n^*,n}(\delta_\infty).$$

Similarly, to show (29), it suffices to show that $V_{p_n^*,n}(\delta_0) \geq V_{p_n^*,n}(\delta_\Delta)$ for all $\Delta \in (0, \infty)$, which we have established in (23). □

Proof of Lemma 5. Let $\hat{V}_{p,n}(\delta_t)$ denote the value from δ_t under state (p, n) (i.e., under common belief p) assuming the firm is informed and thus holds belief 1. To prove the above statement, it suffices to show

$$\hat{V}_{p,n}(\delta_0) \geq \hat{V}_{p,n}(\delta_\Delta) \text{ for all } t \in [0, \infty]. \tag{30}$$

Fix an n and assume by induction that the statement holds for all $m > n$. First, suppose $k_n < \beta$ and $p \leq p_n^*$. Then, by (P), $\alpha_n(p) = 1$. So, $\hat{V}_{p,n}(\delta_0) = k_n$. This is the maximum payoff that can be achieved for all $m \geq n$, and thus (30) holds. Next, suppose $k_N \geq \beta$. In this case, by (P) and the inductive assumption, $\hat{V}_{p,n}(\delta_t) = \beta$ for all $t \in [0, \infty)$. Meanwhile, $\hat{V}_{p,n}(\delta_\infty) < \beta$. Thus, (30) holds. Finally, suppose $k_n < \beta$ or $p > p_n^*$. Then, for all $\Delta \in [0, \infty)$,

$$\hat{V}_{p,n}(\delta_\Delta) = (N-n) \int_0^\Delta k_{n+1} \alpha_{n+1}(p^i(s)) d\Psi^{-i}(s) + (1 - (N-n)\Psi^{-i}(\Delta)) \alpha_n(p(\Delta)),$$

where $\Psi^{-i}(\Delta) = \int_0^\infty (1 - F_{p,n}(s))^{N-n} e^{-\lambda(N-n)s} d((F_{p,n}(s) - 1)e^{-\lambda s})$. Differentiating, we have:

$$\frac{d}{d\Delta}\hat{V}_{p,n}(\delta_\Delta) = [k_{n+1}\alpha_{n+1}(p^i(\Delta)) - k_n\alpha_n(p(\Delta))]\Psi^{-i'}(\Delta)(N-n) + [1-(N-n)\Psi^{-i}(\Delta)]p'(\Delta)\alpha_n'(p(\Delta)),$$

where $\Psi^{-i'}(\Delta) = \lambda(1 - F_{p,n}(\Delta))^{N-n} e^{-\lambda\Delta(N-n)} \frac{p(\Delta)}{\alpha_n(p(\Delta))}$. Substituting, we have

$$\frac{d}{d\Delta}\hat{V}_{p,n}(\delta_\Delta) = K[k_{n+1}\alpha_{n+1}(p^i(\Delta)) - V_{p^i(\Delta),n+1} - \beta(1 - \alpha_n(p))(1-p)],$$

where $K > 0$ is a constant. Now, note that

$$V_{p^i(\Delta),n+1} \geq V_{p^i(\Delta),n+1}(\delta_0) = k_{n+1}\alpha_{n=1}(p^i(\Delta)) - (1 - p(\Delta))(1 - \alpha_n(p(\Delta))\beta.$$

Thus, $\frac{d}{d\Delta}\hat{V}_{p,n}(\delta_\Delta) \leq 0$, and therefore $\hat{V}_{p,n}(\delta_0) \geq \hat{V}_{p,n}(\delta_\Delta)$ for all $\Delta > 0$. It remains to show that $\hat{V}_{p,n}(\delta_0) > \hat{V}_{p,n}(\delta_\infty)$.

$$\hat{V}_{p,n}(\delta_\infty) = (N-n)\int_0^\infty k_n\alpha_{n+1}(p^i(s))d\Psi^{-i}(s) = \lim_{\Delta\to\infty}\hat{V}_{p,n}(\delta_\Delta) \leq \hat{V}_{p,n}(\delta_0).$$

□

Proofs of Welfare Comparison Results

Before proceeding with the proof of Proposition 6, I establish a useful lemma. This establishes that equilibrium credibility is continuous in the parameters.

Lemma 7. *The equilibrium $\alpha_n(p)$ is continuous in the parameters $\{k_n\}_{n=1}^N$ and β.*

Proof. Let us recall that (ODE) is Lipschitz continuous in $\alpha_n(p)$ on $[x, 1+\varepsilon]$ for any $x > 0$ and some $\varepsilon > 0$. It thus follows from Theorem 12.1 in Agarwal, O'Regan, et al. (2008) that $\alpha_n(p)$ is continuous in $\{k_n\}_{n=1}^N$ and β for any n. □

Proof of Proposition 6. I begin by proving there exists such a $\bar{k}$. Fix a k_1, β, λ, p_0 and $N > 1$. I begin by showing that welfare is lower under competition with winner-takes-all payoffs. Let W^C (W^{NC}) denote welfare under competition and under the competition-free benchmark, respectively. I show that if $k_2 = 0$, $W^C = W^{NC}$ if there is no faking in equilibrium and $W^C < W^{NC}$ if faking occurs in equilibrium.

Let α_n^C and α denote equilibrium credibility under competition and the no-competition benchmark, respectively. It follows from Proposition 1 and Proposition 2 that

$$\alpha_1^C(p) \leq \alpha(p) \text{ for all } p, \tag{31}$$

where the inequality holds strictly when $\alpha_1^C(p) < 1$. Now, let $F^C(t)$ [$F(t)$] denote the distribution of the time of the first report in equilibrium under competition and the no-competition benchmark, respectively. Let $f^C(t)$ [$f(t)$] denote the associated density function and $h^C(t)$ [$h(t)$] the associated hazard rate.

Define $x(t) \equiv h^C(t) - h(t)$. It follows from (31) that $x(t) \geq 0$ for all t and $x(t) > 0$ when $\alpha_1^C(p_0(t)) < 1$. Now define the PDF of report times generated by a non-homogeneous Poisson process with hazard rate $x(t)$ as $g(t)$:

$$g(t) \equiv x(t)e^{-\int_0^t x(s)ds},$$

and let $G(t)$ denote the associated CDF. It follows that

$$\begin{aligned}
W^C = &\int_0^\infty [\alpha_1^C(p_0(t))[4\alpha(p_0(t)) - 3\alpha_1^C(p_0(t))](1 - G(t))f(t)dt \\
&- \int_0^\infty \alpha_1^C(p_0(t))^2[\lim_{s\to\infty} F(s) - F(t)]g(t)dt - \int_0^\infty \alpha_1^C(p_0(t))^2[1 - \lim_{s\to\infty} F(s)]g(t)dt \\
W^{NC} = &\int_0^\infty \alpha(p_0(t))^2(1 - G(t))f(t)dt + \int_0^\infty \alpha(p_0(t))^2[\lim_{s\to\infty} F(s) - F(t)]g(t)dt.
\end{aligned}$$

If $\alpha_1^C(p_0(t)) = 1$ for all $t \geq 0$, $G(t) = 0$ for all t and thus $W^{NC} = W^C$. Meanwhile, if $\alpha_1^C(p_0(t)) < 1$ for some $t \geq 0$, $W^{NC} > W^C$.

First consider the case where there is positive faking under $k_2 = 0$. It follows from the above comparison of W^{NC} and W^C, along with Lemma 7 that there exists $\overline{k}$ such that if $k_2 < \overline{k}$, $W^C < W^{NC}$. Next, consider the case where there is no faking under $k_2 = 0$. It follows from Proposition 1 that $\beta > k_1$ and $p_0 \leq p^* \equiv \min\{\frac{k_1-\beta}{\frac{k_1}{N}-\beta}, 1\}$, where $\beta \neq k_1$ by assumption. It further follows from Proposition 1 that there exists $\overline{k} > 0$ such that if $k_2 < \overline{k}$, $\alpha_1^C(p_0(t)) = 1$ for all $t > 0$. Thus, conditional on a first report being made, the common belief becomes $p = 1$ and a second report will thus immediately be made (and will always be fake). Thus, for all such k_2 where $k_2 > 0$, $W^C < W^{NC}$.

Now, I show that there exists such a $\underline{k}$. I begin by showing that when $k_n = k_1$

for all $n > 1$, $W^C = W^{NC}$. Let P_n^C denote the probability that an n^{th} report is made under competition, and P^{NC} the probability that a report is made under the no-competition benchmark. By Proposition 4,

$$W^{NC} = P^{NC}[\frac{\beta}{k_1}]^2 \qquad W^C = \frac{1}{N}\sum_{n=1}^{N} P_n^C[\frac{\beta}{k_1}]^2.$$

It also follows from Proposition 4 that $P_1^C = P^{NC}$ and that $P_n^C < P^{NC}$ for all $n > 1$. Thus, $W^C < W^{NC}$. It then follows from Lemma 7 that there exists $\underline{k} \in (0, k_1)$ such that if $k_n > \underline{k}$ for all $n > 1$, $W^C < W^{NC}$.

□

Proofs of Comparative Statics Results

Proof of Comparative Static 1. First, we establish part (a). Fix all other parameters and let $0 < \beta < \tilde{\beta}$. Let α and $\tilde{\alpha}$ denote the equilibrium credibility functions under β and $\tilde{\beta}$, respectively. Fix an n and assume inductively that the proposition holds for $n+1$ if $n < N$. Note that for any (p, n) and t, $p(t)$ will be the same under β and $\tilde{\beta}$. Thus to show the above claim, it suffices to show that for any p, $\alpha_n(p)$ is weakly increasing in β, and strictly so whenever $\alpha_n(p) < 1$.

We begin by showing that $\alpha_n(p) = 1$ implies that $\tilde{\alpha}_n(p) = 1$. First, consider the case where $n = N$. By Proposition 2, $\alpha_N(p) = 1$ implies that $k_N \leq \beta$. Thus, $k_N < \tilde{\beta}$, which by Proposition 1 implies that $\tilde{\alpha}_N(p) = 1$. Next, consider the case where $n < N$, and assume $\alpha_n(p) = 1$. By Proposition 1, this implies that $k_n < \beta$ and $p \leq p_n^* \equiv \frac{\beta - k_n}{\beta - k_n/n}$. Further note that

$$\tilde{p}_n^* \equiv \frac{\tilde{\beta} - k_n}{\tilde{\beta} - k_n/n} > \frac{\beta - k_n}{\beta - k_n/n} \equiv p_n^*.$$

Thus, $k_n < \tilde{\beta}$ and $p < \tilde{p}_n^*$, which by Proposition 1 implies $\tilde{\alpha}_n(p) = 1$.

Now, suppose that $\alpha_n(p) < 1$. We wish to show that $\tilde{\alpha}_n(p) > \alpha_n(p)$. Suppose by contradiction that $\tilde{\alpha}_n(p) \leq \alpha_n(p)$. It follows from Proposition 2 that if $k_n > \tilde{\beta}$,

$$\lim_{q \to 0+} \alpha_n(q) = \beta/k_n < \tilde{\beta}/k_n = \lim_{q \to 0+} \tilde{\alpha}_n(q).$$

Meanwhile, if $k_n \leq \tilde{\beta}$. $\lim_{q \to \tilde{p}_n^*+} \alpha_n(q) < 1 = \lim_{q \to \tilde{p}_n^*+} \tilde{\alpha}_n(q)$. To see why the latter must hold, first consider the case where $n = N$. It follows from Lemma 4 that $\tilde{\alpha}_n(q) = 1$ for all q. Meanwhile, it follows again from Proposition 2 that $\alpha_N(q)$ is constant in q, and because $\alpha_N(p) < 1$, $\lim_{q \to \tilde{p}_n^*+} \alpha_N(q) < 1$. In the case where $n < N$, because $p_n^* < \tilde{p}_n^*$, it follows from Proposition 1 that $\alpha_n(\tilde{p}_n^*) < 1$.

Thus, we have that both when $k_n > \tilde{\beta}$ and when $k_n \leq \tilde{\beta}$, there exists some $\hat{p} < p$ such that $\tilde{\alpha}_n(\hat{p}) > \alpha_n(\hat{p})$ and $\tilde{\alpha}_n, \alpha_n$ satisfy (ODE) on $[\hat{p}, p]$, for their respective value of β. Thus, there exists a $q \in [\hat{p}, p]$ such that $\alpha_n(q) = \tilde{\alpha}_n(q)$ and $\alpha_n'(q) \geq \tilde{\alpha}_n'(q)$. It follows from (ODE) that in order for the above two conditions to hold, it must be that

$$X \equiv (\beta - \tilde{\beta})(\frac{1 - \alpha_n(q)}{\alpha_n(q)})(1 - q) + \frac{V_{q^i,n+1} - \tilde{V}_{q^i,n+1}}{\alpha_n(q)} \geq 0. \tag{32}$$

where V and $\tilde{V}$ denote the value functions under β and $\tilde{\beta}$, respectively. First consider the case where $n = N$. Then $V_{q^i,n+1} = V_{\tilde{q}^i,n+1} = 0$, and thus $X < 0$, contradicting (32).

Next, consider the case where $n < N$. First suppose that $\alpha_{n+1}(q^i) = 1$. It follows from the inductive assumption that $\tilde{\alpha}_{n+1}(q^i) = 1$. Thus, by Lemma 4, $V_{q^i,n+1} = \frac{k_{n+1}q^i}{N-n} = \tilde{V}_{q^i,n+1}$. Again this implies that $X < 0$, contradicting (32). Now, suppose that $\alpha_{n+1}(q^i) < 1$. It then follows from Lemma 2 that $V_{q^i,n+1} = k_{n+1}\alpha_{n+1}(q^i) - \beta(1 - q^i)$. Furthermore,

$$\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_0) = k_{n+1}\tilde{\alpha}_{n+1}(q^i) - \tilde{\beta}(1 - q^i).$$

Thus, recalling that $q^i = \alpha_{n+1}(q) + (1 - \alpha_{n+1}(q))q$, we have

$$V_{q^i,n+1} - \tilde{V}_{q^i,n+1} \leq k_{n+1}(\alpha_{n+1}(q^i) - \tilde{\alpha}_{n+1}(q^i)).$$

Substituting this into the above expression for X, we obtain

$$X \leq \frac{k_{n+1}(\alpha_{n+1}(q^i) - \tilde{\alpha}_{n+1}(q^i))}{\alpha_n(q)} < 0.$$

where the strict inequality follows from the inductive assumption that $\alpha_{n+1}(q^i) < \tilde{\alpha}_{n+1}(q^i))$. Again, this is a contradiction of (32). □

Proof of Corollary 4. Fix all other parameters and let $0 < \beta < \tilde{\beta}$. Assume a winner-takes-all setting ($k_n = 0$ for $n > 1$). Let α (V) and $\tilde{\alpha}$ ($\tilde{V}$) denote the equilibrium credibility (value function) under β and $\tilde{\beta}$, respectively. We want to show that $V_{p_0,1} \leq \tilde{V}_{p_0,1}$ and $V_{p_0,1} < \tilde{V}_{p_0,1}$ when $\alpha_1(p_0) < 1$.

First, suppose $\alpha_1(p_0) = 1$. Because firms are truthful in this case,

$$V_{p_0,1} = V_{p_0,1}(\delta_\infty) = \frac{k_1}{N},$$

where the exact same equality holds under $\tilde{\beta}$. Thus, $V_{p_0,1} = \tilde{V}_{p_0,1}$.

Next, suppose $\alpha_1(p_0) < 1$. It follows from Lemma 2 that $V_{p_0,1} = V_{p_0,1}(\delta_\infty)$ and $\tilde{V}_{p_0,1} = \tilde{V}_{p_0,1}(\delta_\infty)$. Now, note that

$$V_{p_0,1}(\delta_\infty) = \int_0^\infty k_1\alpha_1(p_0(s))\psi^i(s)ds \text{ and } \tilde{V}_{p_0,1}(\delta_\infty) = \int_0^\infty k_1\tilde{\alpha}_1(p_0(s))\tilde{\psi}^i(s)ds, \text{ where}$$

$\psi^i(s) = p\lambda e^{-\lambda sN}(1 - F_{p_0,1}(s))^{N-1}$ and $\tilde{\psi}^i(s) = p\lambda e^{-\lambda sN}(1 - \tilde{F}_{p_0,1}(s))^{N-1}$, and F ($\tilde{F}$) is the equilibrium strategy under β ($\tilde{\beta}$). Now, note by Comparative Static 1 that

$$\alpha_1(p_0(s)) \leq \tilde{\alpha}_1(p_0(s)), \tag{33}$$

where the in equality holds strictly for some interval of s. Likewise,

$$b_1(p_0(s)) \geq \tilde{b}_1(p_0(s)),$$

where the in equality holds strictly for some interval of s. This implies

$$F_1(p_0(s)) > \tilde{F}_1(p_0(s)), \text{ for all } s > 0.$$

This, combined with (33), implies that $V_{p_0,1} < \tilde{V}_{p_0,1}$.

□

Proof of Comparative Static 2. Let $\tilde{\lambda} > \lambda > 0$, and let α, $\tilde{\alpha}$ denote the equilibria under λ and $\tilde{\lambda}$, respectively, fixing all other parameters. We begin by showing that $\tilde{\alpha}_n(p) = \alpha_n(p)$ for any p and n. Fix an n and assume inductively that if $n < N$, $\alpha_{n+1}(p) = \tilde{\alpha}_{n+1}(p)$ for all p on-path. Let V, $\tilde{V}$ denote the value functions under the equilibria associated with λ and $\tilde{\lambda}$, respectively. Note that $V_{p,n+1} = \tilde{V}_{p,n+1}$ for all p on-path. In the case where $n = N$, $V_{p,n+1} = \tilde{V}_{p,n+1} = 0$, and thus this holds trivially. In the case where $n < N$, this follows from the inductive assumption.

By Lemma 4, α_n and $\tilde{\alpha}_n$ must both be a solution to (P) at all (p, n) on-path, which does not depend on λ. By Theorem 1, the solution to (P) is unique, and thus $\alpha_n(p) = \tilde{\alpha}_n(p)$ at all (p, n) on-path. Now fixing any p and n, let $p(t)$ and $\tilde{p}(t)$ denote the common beliefs under λ and $\tilde{\lambda}$, respectively. It follows from (2) that $p(t) > \tilde{p}(t)$ for all $t > 0$. Thus, because $\alpha_n(p)$ and $\tilde{\alpha}_n(p)$ are both weakly decreasing in p (Proposition 3), it follows that $\alpha_n(p(t)) \leq \tilde{\alpha}_n(p(t))$. Furthermore, since $\tilde{\alpha}(p)$ is strictly decreasing in p (Proposition 3) whenever $\alpha_n(p) < 1$ and $k_N > \beta$, it follows that $\alpha_n(p(t)) < \alpha_n(\tilde{p(t)})$. □

Proof of Comparative Static 3. Let α and $\tilde{\alpha}$ denote the equilibria under N and $N+1$ firms, respectively, fixing all other parameters. We begin by showing that for all p, $\alpha_n(p) \geq \tilde{\alpha}_n(p)$, and $\alpha_n(p) > \tilde{\alpha}_n(p)$ when $\alpha_n(p) < 1$. To this end, fix an $n \in \{1, ..., N\}$ and assume inductively that the claim holds for $n+1$ whenever $n < N$. We begin by showing that $\tilde{\alpha}_n(p) = 1$ implies that $\alpha_n(p) = 1$. Suppose that $\tilde{\alpha}_n(p) = 1$. By Proposition 1, $\beta > k_n$ and $p < \tilde{p}_n^* \equiv \frac{\beta - k_n}{\beta - k_n/(N+1-n)}$. Because $p_n^* \equiv \frac{\beta - k_n}{\beta - k_n/(N-n)} > \tilde{p}_n^*$, it follows from Proposition 1 that $\alpha_n(p) = 1$.

Now consider the case where $\tilde{\alpha}_n(p) < 1$. We wish to show that $\tilde{\alpha}_n(p) < \alpha_n(p)$. To this end, we begin by making the following observation:

$$\text{If } \alpha_n \text{ and } \tilde{\alpha}_n \text{ both satisfy (ODE) at } q, \text{ and } \alpha_n(q) = \tilde{\alpha}_n(q), \text{ then } \alpha_n'(q) > \tilde{\alpha}_n'(q). \tag{34}$$

Let us now establish this. Note first that for α_n and $\tilde{\alpha}_n$ to both satisfy (ODE) at q, given that $\alpha_n(q) = \tilde{\alpha}_n(q)$, the following must hold:

$$\alpha_n'(q) = \frac{-1}{k_n(1-q)\alpha_n(q)} \frac{N-n}{N-n+1} (k_n \alpha_n(q) - V_{q^i, n+1} - \beta(1 - \alpha_n(q))(1-q))$$

$$\tilde{\alpha}_n'(q) = \frac{-1}{k_n(1-q)\alpha_n(q)} \frac{N-n+1}{N-n+2} (k_n \alpha_n(q) - \tilde{V}_{q^i, n+1} - \beta(1 - \alpha_n(q))(1-q)),$$

where V and $\tilde{V}$ denote the value functions under the equilibria with N and $N+1$ total firms, respectively. Note that if $n = N$, $\alpha_n'(q) = 0$. Meanwhile, by Proposition 3, $\tilde{\alpha}_n'(q) < 0$. Thus, $\tilde{\alpha}_n'(q) < \alpha_n(q)$ must hold. Next, consider the case where $n < N$. We begin by observing that $V_{q^i, n+1} > \tilde{V}_{q^i, n+1}$. To see why this must hold, first consider the case where $\tilde{\alpha}_{n+1}(q^i) = 1$. It then follows from the inductive assumption that $\alpha_n(q^i) = 1$. Then, by Lemma 4,

$$\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_\infty) = \frac{k_{n+1}q^i}{N-n} < \frac{k_{n+1}q^i}{N-n-1} = V_{q^i,n+1}(\delta_\infty) = V_{q^i,n+1}.$$

Next, consider the case where $\tilde{\alpha}_n(q^i) < 1$. In this case, it follows from Lemma 2 that

$$\begin{aligned}\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_0) = k_{n+1}\tilde{\alpha}_{n+1}(q^i) - \beta(1-q^i) < k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i)\\ = V_{q^i,n+1}(\delta_0) \le V_{q^i,n+1},\end{aligned}$$

where the strict inequality follows from the inductive assumption made above. Examining the two ODEs listed above, since by Proposition 3, $\alpha_n'(q) \le 0$, it follows that $\tilde{\alpha}_n'(q) < \alpha_n'(q)$.

Now, assume by contradiction that $\alpha_n(p) \le \tilde{\alpha}_n(p)$. We begin by showing that there exists a $q^* < p$ such that $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$. First consider the case where $k_n \ge \beta$. Then, by Proposition 2,

$$\lim_{q \to 0+} \alpha_n(q) = \lim_{q \to 0+} \tilde{\alpha}_n(q) = \frac{\beta}{k_n}.$$

Then, by the continuous differentiability of α_n and $\tilde{\alpha}_n$ on $(0,p)$, it follows from Equation 34 that for some $q^* < p$ sufficiently small $\alpha_n(q^*) > \tilde{\alpha}_n(q^*)$. Next, consider the case where $k_n < \beta$, and let $p_n^* \equiv \frac{\beta - k_n}{\beta/(N-n+1)-k_n}$. Note by Proposition 1 that $\alpha_n(p_n^*) = 1$. Meanwhile, because $p_n^* < \tilde{p}_n^* \equiv \frac{\beta - k_n}{\beta/(N-n+2)-k_n}$, it follows from Proposition 1 that $\tilde{\alpha}_n(p_n^*) < 1$, and thus, we have for $q^* = p_n^*$, $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$.

Since $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$ and $\tilde{\alpha}_n(p) \ge \alpha_n(p)$, by the continuous differentiability of α on $[q^*, p]$, there must exist some $q \in (q^*, p]$ such that $\alpha_n(q) = \tilde{\alpha}_n(q)$ and $\alpha_n'(q) \le \tilde{\alpha}_n'(q)$. However, this is a contradiction of (34).

Now fixing any p and n, let $p(t)$ and $\tilde{p}(t)$ denote the common beliefs under N and $N+1$ firms, respectively. We wish to show that on some interval $[0, \bar{t}]$, where $\bar{t} > 0$, $\alpha_n(p(t)) \ge \tilde{\alpha}_n(\tilde{p}(t))$ is weakly increasing in t, and strictly so whenever $\alpha_n(p(t)) < 1$. First consider the case where $\alpha_n(p(t)) = 1$. In this case, the statement holds trivially. Next, consider the case where $\alpha_n(p) < 1$. It follows from the above that $\alpha_n(p) > \tilde{\alpha}_n(p)$. Now note that it follows from (2) that $\lim_{t \to 0+} p(t) - \tilde{p}(t) = 0$. Since $\alpha_n(p(t))$ and $\tilde{\alpha}_n(\tilde{p}(t))$ are both continuous in t (Lemma 3), it follows that for some $\bar{t} > 0$, $\alpha_n(p(t)) > \tilde{\alpha}_n(\tilde{p}(t))$ for all $t \in [0, \bar{t}]$. □

Extension: heterogeneous ability

I now consider an extension in which firms have heterogeneous learning abilities. This will shed light on how a firm's credibility correlates with its ability in equilibrium.

The extended model is identical to the model above except for three changes. First, rather than assuming that each firm is endowed with the same ability λ, each firm i is endowed with a firm-specific ability λ^i, which is common knowledge. Second, for tractability, I restrict attention to a winner-takes-all setting: i.e., I assume $k_n = 0$ for all $n > 1$. Finally, I relax the equilibrium symmetry assumption. Accordingly, let α^i denote the credibility of firm i.

I obtain an intuitive result: firms with higher ability are more credible in equilibrium.

Proposition 7. *For all (i, j) such that $\lambda^i < \lambda^j$, $\alpha_1^i(p(t)) \leq \alpha_1^j(p(t))$. Furthermore, this inequality is strict whenever $\alpha_1^i(p(t)) < 1$.*

Proposition 7 states that regardless of when a report is made, a firm with higher ability is weakly more credible, and strictly so whenever firms are not fully truthful. Let us consider why this correlation arises. First, note that high ability firms are able to confirm a story more quickly and thus, all else equal, pose a greater preemptive threat in equilibrium. This in turn implies that in comparison to a high-ability firm, a low-ability firm faces a greater preemptive threat. Thus, the low-ability firm finds immediate faking more advantageous. In light of this, the firms' credibilities must adjust in such a way to preserve their respective indifference conditions. This is achieved endogenously by means of a lower credibility for the low-ability firm, which ensures that it has less to gain from faking.

Proofs: heterogeneous learning ability

Here, we consider the extended model presented in above. The objective is to establish Proposition 7. This proof will require extending certain results established in the baseline model to the extended model. Regarding Lemmas 1-4, I will take for granted that these hold under the extended model. Formal proofs of this are omitted as all proofs presented under the baseline model will also apply to the extended setting.

Next, I establish that Proposition 1 holds under the extended model. This claim is presented as Proposition 1′. In the analysis below, I let $V^i_{p,n}$ denote firm i's value.

Proposition 1′. *For all i, there exists a $p^{i*} \in (0, 1]$ such that at any p on-path, $\alpha^i_1(p) = 1$ if and only if the following two conditions hold:*

1. $k_1 \leq \beta$
2. $p \leq p^{i*}$

Furthermore, $p^{j*} > p^{i*}$ whenever $\lambda^j > \lambda^i$.

Proof. Fix an i. Suppose that $k_1 \leq \beta$. By identical reasoning as Proposition 1, for all $q < \frac{\beta - k_1}{k_1}$, $\alpha^i_1(q) = 1$. Let

$$p^{i*} \equiv sup\{p | \alpha^i_1(p) = 1 \text{ for all } q < p\}.$$

It follows by definition that $\alpha^i_1(p) = 1$ for all $p \leq p^{i*}_1$.

Next, we will show that $\alpha^i_1(q) < 1$ whenever $k_1 > \beta$ or $p > p^{i*}_1$. Suppose not by contradiction. First, consider the case where $k_1 > \beta$ and $\alpha^i_1(p) = 1$ for some p. Then we have that

$$V^i_{p,1}(\delta_0) = k_1 p + (k_1 - \beta)(1 - p) > k_1 p \geq V^i_{p,1}(\delta_\infty)$$

Thus, i can profitably deviate at p. Contradiction. Next, consider the case where $q > p^{i*}_1$ and $\alpha^i_1(p) = 1$. In this case, a contradiction follows from identical reasoning to what is presented in Proposition 1.

Finally, we show that $p^{j*} > p^{i*}$ whenever $\lambda^j > \lambda^i$. Suppose by contradiction that $p^{j*} \leq p^{i*}$. Note that because j is truth telling at $(p^{j*}, n = 1)$, $V^j_{p^{j*},1}(\delta_\infty) \geq V^j_{p^{j*},1}(\delta_0)$. Furthermore, because $p^{j*} \leq p^{i*}$, i is also truthful at $(p^{j*}_n, n = 1)$. Thus,

$$V^j_{p^{j*}_1,1}(\delta_0) = V^i_{p^{j*}_1,1}(\delta_\infty) = k_1 - \beta(1 - p).$$

Now, note that because $\lambda^j > \lambda^i$,

$$V^j_{p^{j*}_1,1}(\delta_\infty) > V^i_{p^{j*}_1,1}(\delta_\infty).$$

Combining these inequalities we have $V^i_{p^{j*}_1,1}(\delta_\infty) < V^i_{p^{j*}_1,1}(\delta_0)$. However, because $\alpha^i_1(p^{j*}) = 1$, $V^j_{p^{j*}_n,1} = V^j_{p^{j*}_n,1}(\delta_\infty)$. Contradiction. □

Next, we extend Proposition 2 to this setting. Note this entails deriving an ODE that applies to this extended model, (ODE-i).

Proposition 2′. *In equilibrium, for any* p *on-path, if* $k_1 \geq \beta$ *or* $p > p^{i*}$*, then the following must be satisfied:*

$$\alpha_1^{i\prime}(p) = -\frac{\beta}{k_1}\left(\frac{\sum_{j\neq i}\lambda^j}{\sum_j \lambda^j}\right) - \frac{\sum_{j\neq i}\frac{\lambda^j}{\alpha_1^j(p)}}{\sum_j \lambda^j(1-p)}[\alpha^i(p) - \frac{\beta}{k_1}(1-p)]. \qquad \text{(ODE-}i\text{)}$$

In addition, $\lim_{p\to 0+}\alpha_1^i(p) = \beta/k_1$ *must hold if* $k_1 > \beta$*, and* $\lim_{p\to p^{i*}+}\alpha_1^i(p) = 1$ *if* $k_1 \leq \beta$*.*

Proof. Let us first establish that (ODE-i) must hold under the conditions specified.

When $k_1 \geq \beta$ or $p > p^{i*}$, it follows from Proposition 1′ that $\alpha_1^i(p(t)) < 1$. It then follows from Lemma 2 that there exists an $\varepsilon > 0$ such that for all $\Delta \in (0, \varepsilon)$,

$$\frac{V_{p,1}^i(\delta_\Delta) - V_{p,1}^i(\delta_0)}{\Delta} = 0.$$

Recall that $V_{p,1}^i(\delta_0) = k_1\alpha_1^i(p) - \beta(1-p)$. Meanwhile,

$$V_{p,1}^i(\delta_\Delta) = \int_0^\Delta k_1\alpha_1^i(p(s))\Psi^i(s)ds + (1 - \sum_j \lim_{s\to\Delta-}\Psi^j(s))[k_1\alpha_1(p(\Delta)) - \beta(1-p(\Delta))],$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play the equilibrium strategy $F_{p,1}^j$. Specifically, for all $s > 0$,

$$\Psi^i(s) = p\lambda^i \int_0^s e^{-\sum_j \lambda^j r}\prod_{j\neq i}(1 - F_{p,1}^i(r)))dr$$

and for $j \neq i$,

$$\Psi^j(s) = p\int_0^s e^{-\sum_{k\neq j}\lambda^k r}\prod_{k\neq i\neq j}(1 - F_{p,1}^k(r))d(-e^{-\lambda^j r}(1 - F_{p,1}^j(r)))$$
$$+(1-p)\int_0^s \prod_{k\neq i\neq j}(1 - F_{p,1}^k(r))dF_{p,1}^j(r).$$

Substituting these two expressions into the above equation for $V_{p,1}^i(\delta_\Delta)$ and following the same sequence of steps as in the proof of Proposition 2 yields

(ODE-i).

The two limit conditions are established by the same reasoning presented in the proof of Proposition 2. □

Proof of Proposition 7. Fix any (i, j) such that $\lambda^i > \lambda^j$. We want to show that $\alpha_1^i(p(t)) \leq \alpha_1^j(p(t))$ and that $\alpha_1^i(p(t)) < \alpha_1^j(p(t))$ whenever $\alpha_1^i(p(t)) < 1$. First suppose $\alpha_1^i(p) = 1$. In this case, $\alpha_1^i(p) \geq \alpha_1^j(p)$ is trivially satisfied.

Next, suppose $\alpha_1^i(p) < 1$. We want to show that $\alpha_1^i(p) > \alpha_1^j(p)$. Suppose by contradiction that $\alpha_1^i(p) \leq \alpha_1^j(p)$. First consider the case where $k_1 < \beta$. Then, let

$$q^* \equiv \inf\{q | \alpha_1^j(p) < 1 \text{ and } \alpha_1^j(p) < \alpha_1^i(p)\}.$$

Because the α_1^i are continuous, it follows from Proposition 1′, and the assumption that $\alpha_1^i(p) \leq \alpha_1^j(p)$, that $q^* < p$ exists. Again, by continuity, $\alpha_1^j(q^*) = \alpha_1^i(q^*)$. It then follows from (ODE-i) that $\alpha_1^{j\prime}(q^*) < \alpha_1^{i\prime}(q^*)$. But this implies that for some $q > q^*$, $\alpha_1^j(q^*) > \alpha_1^i(q^*)$. Contradiction.

Next, consider the case where $k_1 \geq \beta$. Recall by Proposition 2′ that $\lim_{p \to 0+} \alpha_1^i(p) = \lim_{p \to 0+} \alpha_1^j(p)$. Thus, there exists some $q \in (0, p]$ such that $\alpha_1^i(p) \leq \alpha_1^j(p)$ and $\alpha_1^{i\prime}(p) \leq \alpha_1^{j\prime}(p)$. However, it again follows from (ODE-i) that $\alpha_1^{i\prime}(p) > \alpha_1^{j\prime}(p)$. Contradiction. □

Alternative payoff specifications

Here, I consider the equilibrium under two alternative payoff specifications for the firm. First, I consider the case where the firms market share from reporting a story is a function of its perceived accuracy (rather than credibility). I show that the copycat effect occurs under this alternative specification as well. Second, I consider the case where the firm's market share does not depend on consumer beliefs, and show that the copycat effect cannot occur in this case.

Alternative specification 1: Perceived accuracy

Under this alternative specification, let us assume that the firm's payoff from making a report is given by

$$k_n \tilde{p} - \beta \mathbb{I}(\theta = 0), \tag{35}$$

where $\tilde{p}$ is the consumer's belief that $\theta = 1$ immediately following the report. I otherwise maintain all assumptions of the baseline model. I now make two observations about the equilibrium under this alternative specification:

1. For a report that is made under state (p, n):

$$\tilde{p} = \alpha_n(p) + (1 - \alpha_n(p))p,$$

where $\alpha_n(p)$ is the credibility of the report as previously defined.

2. Lemma 1 does not necessarily hold, i.e., there may exist point masses in the distribution of faking. This is due to the fact that, unlike the baseline specification, a firm will earn a strictly positive market share $k_n p$ in equilibrium from making a report with strictly positive probability at a given history.

In light of point 2 above, I must first augment the definition of the copycat effect to take into account the possibility that there is a point mass in faking. Formally, letting $q_n(p) \equiv F_{p,n}(0)$, we say a report at (p, n) triggers the *copycat effect* if:

$$q_{n+1}(\tilde{p}) - q_n(p) > 0 \text{ and } \tilde{p} \neq 1 \tag{36}$$

or

$$q_{n+1}(\tilde{p}) = 0 \text{ and } b_{n+1}(\tilde{p}) - b_n(p) > 0.$$

(36) takes into account the possibility that there is a point mass in faking: in this case the copycat effect occurs if a report triggers an increase in the point mass of faking (assuming the common belief does not increase to 1, in which such an increase trivially occurs by (SC)).

Before proceeding, I show that if the common belief is sufficiently low, there cannot be point masses in faking.

Lemma 8. *For any n, if $p < \overline{p} \equiv \frac{\beta}{k_n + \beta}$, then $q_n(p) = 0$.*

Proof. Fix an n and suppose that $p < \overline{p}$. Suppose by contradiction that $q_n(p) > 0$. By the definition of $\overline{p}$:

$$V_{p,n}(\delta_0) = k_n(\tilde{p}) - \beta(1 - p) < 0.$$

It thus follows from the same reasoning presented in the proof of Lemma 1 that (20) is a profitable deviation. □

I now show that the copycat effect occurs in this case. Namely, I show that Corollary 3 holds under this alternative specification.

Proposition 8. *Corollary 3 holds under the alternative payoff specification* (35)*.*

Proof. First, note by Lemma 8 there exists a $\overline{p}$ such that for all $p < \overline{p}$, if $q_{n+1}(\tilde{p}) > 0$ a report triggers the copycat effect. Now, I claim that there exists a $\overline{p}'$ such that for all $p < \overline{p}'$, if $q_{n+1}(\tilde{p}) = 0$, then $b_{n+1}(\tilde{p}) - b_n(p) > 0$. It suffices to show that $\lim_{p\to 0+} b_{n+1}(\tilde{p}) - b_n(p) > 0$. We proceed in a number of steps:

1. First, we show $\lim_{p\to 0+} b_n(p) = 0$. Suppose not, by contradiction. Then there exists a $\delta > 0$ such that for all $\varepsilon > 0$, there exists a $\hat{p} \in [0, \varepsilon]$ such that $b_n(p) > \delta$. For any such $\hat{p}$,
$$\alpha_n(p) = \frac{\lambda\hat{p}}{\lambda\hat{p} + b_n(p)} < \frac{\lambda\hat{p}}{\lambda\hat{p} + \delta} \equiv \overline{\alpha}(\hat{p}).$$
Now, note that for any given $\hat{p}$
$$V_{\hat{p},n}(\delta_0) < k_n(\hat{p}(1 - \overline{\alpha}(\hat{p})) + \overline{\alpha}(\hat{p})) - \beta(1 - \hat{p}) < k_n\hat{p}[\frac{\delta + \lambda}{\lambda\hat{p} + \delta}] - \beta(1 - \hat{p}).$$
The right-hand side of the above inequality is continuous in $\hat{p}$ and strictly negative when $\hat{p} = 0$. Thus for some ε, for all $\hat{p} \in [0, \varepsilon]$, $V_{\hat{p},n}(\delta_0) < 0$. It thus follows from the same reasoning as that presented in Lemma 6 that there is a profitable deviation.

2. Next, show $\lim_{p\to 0+} \alpha_n(p) = \frac{\beta}{k_n}$. First, note by Lemma 8, it follows from the same reasoning presented in Lemma 2 that for all $p < \overline{p}$, $V_{p,n}(\delta_0) = V_{p,n}(\delta_\infty)$. Furthermore, $\lim_{p\to 0+} V_{p,n}(\delta_\infty) = 0$. Thus,
$$\lim_{p\to 0+} V_{p,n}(\delta_0) = \lim_{p\to 0+} k_n[\alpha_n(p) + (1 - \alpha_n(p))p] - \beta = 0.$$
Thus, $\lim_{p\to 0+} \alpha_n(p) = \frac{k_n}{\beta}$.

Thus, by the continuity of b_{n+1}: $\lim_{p\to 0+}[b_{n+1}(\tilde{p}) - b_n(p)] = b_{n+1}(\beta/k_n) > 0$. □

Alternative specification 2: exogenous payoffs

To demonstrate the role of endogenous payoffs on the equilibrium, I consider an alternative payoff specification where market share is exogenous. Specifically, I assume the firm's payoff is given by the following:

$$k_n - \beta\mathbb{I}(\theta = 0),$$

where $k_1 > k_2 > ... > k_N \geq 0$. Finally, I assume away the knife-edge case where $\beta = k_n$ for any k_n.

I now show that under such a specification, the equilibrium is such that firms either report truthfully or report immediately. Furthermore, while in general there will exist asymmetric equilibria, I will drop the firm's index when referring to their strategy.

Lemma 9. *In any equilibrium for all $p < 1$, $F'_{p,n}(s) = 0$.*

Proof. Suppose by contradiction that the statement does not hold. First, I show that if $F_{p,n}(s) = 0$ for $s = 0$, then $F_{p,n}(s) = 0$ for all $s > 0$. Suppose not, by contradiction. I.e., suppose that in some equilibrium $F_{p,n}(0) = 0$ and $F_{p,n}(s^*) > 0$ for some $s^* > 0$. Now, consider the following alternative strategy:

$$\tilde{F}_{p,n}(s) = \begin{cases} F_{p,n}(s^*) & \text{for } s \in [0, s^*] \\ F_{p,n}(s) & \text{for } s > s^*. \end{cases}$$

It follows that

$$V_{p,n}(\tilde{F}) = V_{p,n}(F) + \sum_{j \neq i} \int_0^{s^*} [k_n - \beta(1 - p^j(s)) - V_{p^j(s),n+1}(F)] d\Psi^j(s).$$

Since the $j \neq i$ must be acting optimally, $\sum_{j \neq i} [k_n - \beta(1 - p^j(s)) - V_{p^j(s),n+1}(F)] > 0$. Thus, $V_{p,n}(\tilde{F}) > V_{p,n}(F)$ and F is not an equilibrium strategy. Contradiction. □

Next, using the above lemma, I show that the copycat effect cannot occur under the exogenous payoff specification. Specifically, a report by a firm can induce an increase in faking only if the report is truthful. I formalize this as Proposition 9.

Proposition 9. *Under exogenous payoffs, the copycat effect never occurs.*

Proof. Fix any (p, n). First note that if $p = 1$, $F_{p,n}(0) = 1$, and hence the copycat effect cannot occur. Now suppose $p < 1$. Let us consider two cases:

1. First, suppose $F_{p,n}(0) = 0$. Then it follows from Lemma 9 that $b_n(p) = 0$, and so $\tilde{p} = 1$. Thus, by definition, the copycat effect does not occur.
2. Suppose $F_{p,n}(0) > 0$. Then $\alpha_n(p) = 0$. Thus $\tilde{p} = p$. By definition, the copycat effect can occur in this case only if $q_{n+1}(p) > q_n(p)$. Assume by contradiction

that this is the case. Therefore, $q_{n+1}(p) > 0$ and $q_n(p) < 1$. Therefore,

$$k_n - \beta(1-p) \leq k_{n+1} - \beta(1-p).$$

Otherwise, $q_{n+1}(p) = 1$ is a profitable deviation. Thus, $k_n \leq k_{n+1}$. Contradiction.

$\square$