Competition and Herding in Breaking News

Sara Shahanaghi*

October 4, 2023

Abstract

I present a dynamic model of breaking news. News firms are rewarded for reporting before their competitors but also for making reports that are credible to consumers. Errors occur when firms fake, reporting a story despite lacking evidence. While errors occur in equilibrium even under a monopoly, competition and observational learning exacerbate errors and give rise to rich dynamics in firm behavior. Competition intensifies faking by engendering a preemptive motive, but the haste-inducing effect of preemption is endogenously mitigated by gradual improvement in report credibility over the course of a news cycle. Meanwhile, observational learning causes existing errors to propagate through the market. This is driven by a *copycat effect*, in which one report triggers an immediate surge in faking by others. This behavior is consistent with herding on the decision to report a story as well as herding on the timing of reports.

*Toulouse School of Economics, sara.shahanaghi@tse-fr.eu. I am grateful of Yeon-Koo Che and Navin Kartik for their guidance. I would like to thank Catherine Bobtcheff, Alessandro Bonatti, Francesc Dilmé, Laura Doval, Alex Frug, Duarte Goncalves, Christian Hellwig, Johannes Hörner, Elliot Lipnowski, Qingmin Liu, Thomas Mariotti, Andrea Prat, Jacopo Perego and participants of various seminars and conferences for helpful feedback.

1. Introduction

What a newspaper needs in its news, in its headlines, and on its editorial page is terseness, humor, descriptive power, satire, originality, good literary style, clever condensation, and accuracy, accuracy, accuracy!

— Joseph Pulitzer

Accuracy is often considered to be the core tenet of news media. This belief is widely held by consumers of news: when asked in a 2018 Pew survey, the majority of respondents listed accuracy as a primary function of news, valuing it over thorough coverage, unbiasedness, and relevance.

Despite this, public perceptions of news accuracy are not favorable. In a 2020 survey, 38% of respondents stated that they go into a news story thinking it will be largely inaccurate. While many factors may contribute to this skepticism, consumers express particular concern about hasty reporting: 53% of respondents state that news breaking too quickly is a major source of errors.

These concerns are supported by a multitude of instances in which news media have made major factual errors. In the immediate aftermath of the 9/11 attacks, cable news stations made several statements that were false: NBC reported an explosion outside the pentagon, CNN reported a fire outside the national mall, and CBS claimed the existence of a car bomb outside the state department. Erroneous reporting has been endemic to terrorist attacks in general, with news media misidentifying perpetrators or other key details of the Boston bombings, Sandy Hook massacre, London bombings, and Oklahoma City bombings. Furthermore, such errors are not limited to terrorist attacks. In 2004, CBS news, under the direction of Dan Rather, published the Killian documents, a collection of memos which called into question George W. Bush's military record. These documents could never be authenticated and were widely believed to be forged. More recent media blunders are ever present: in 2017, ABC news falsely reported that Michael Flynn would testify that Donald Trump had directed him "to make contact with the Russians." In 2019, ABC News headlined its nightly news broadcast with what it claimed to be exclusive footage of the ongoing air strikes on Syria. It was later revealed that this footage was from a machine gun convention in Oklahoma.

While such errors are commonplace, they are also costly to news firms. For one, exposure of errors can be reputationally damaging. This was especially true of the *Rolling Stone* scandal, in which the magazine falsely accused a group of University of Virginia students of sexual assault. Not only was the journalistic failure widely reported by other firms, the error resulted in several publicized lawsuits against the magazine. Furthermore, such errors often lead firms to oust journalists in an apparent effort to protect their reputations. This was evident in the terminations of Dan Rather and Brian Ross —both lead journalists at major news stations—following their respective reporting blunders.

The objective of this paper is twofold. First, I seek to understand why reporting errors are pervasive despite their costliness to firms. In particular, I consider how strategic forces can induce firms to commit errors that are otherwise avoidable. My second objective is to study the dynamics of breaking news. Namely, I ask *when* over the course of a news cycle firms are more susceptible to these strategic forces and thus more prone to erring.

To answer these questions, I present a dynamic model of breaking news. Firms learn privately aboout a story by receiving confirmation that it is true, and must choose if and when to report it. Errors occur when firms fake, i.e. report despite lacking confirmation. Because reports are public, firms also learn by observing the reporting behavior of their opponents. Regarding incentives, firms are penalized for errors but are rewarded for viewership, which hinges on two qualities of the firm's report. First, all else equal, a firm who preempts its rivals enjoys greater viewership. Second, viewership depends on the credibility of the firm's report, i.e. consumers' belief that the story was confirmed before it was reported. Namely, a report is consumed only to the extent that there is trust in the firm's journalistic standards.

I establish existence and uniqueness of an equilibrium. Under this equilibrium, the time at which a firm fakes is distributed continuously. That is, fake reports are made as if they are being generated by a non-homogenous Poisson process. This mixing implies an indifference condition: at any time in which the firm might fake, it must be indifferent between faking and truth telling. This condition in turn implies an ordinary differential equation (ODE) on the arrival rate of fake reports. The equilibrium is characterized by a recursive system of these ODEs, a fact that is central to the analysis.

In equilibrium, errors are strategic responses to three features of the newsroom: a lack of commitment power by firms, competition, and observational learning. Competition and observational learning not only exacerbate faking, but introduce distinct dynamics in firm behavior. I begin by showing that errors can occur even in the absence of competition. In particular, if the cost of error is relatively small —because consumers are less aware or critical of them —even a monopolist will fake. Such errors are driven by a firm's inability to commit to a reporting strategy. Because consumers cannot detect faking, credibility is unaffected by deviations in the firm's reporting strategy. The firm is thus tempted to fake in order to capitalize on its credibility. I substantiate this intuition by showing that a monopolist that can commit to a reporting strategy would never fake, and thus never err.

I then analyze a multi-firm setting, and find that both competition and observational learning can exacerbate errors, but do so in different ways. Competition can incentivize speed by giving rise to a preemptive motive, which makes faking more valuable. To restore indifference, credibility must fall to make faking less valuable, which in equilibrium can only be consistent with more faking. Notably, this preemptive motive is not an artifact of the firm's payoff function, but rather an equilibium phenomenon. In fact, whenever the cost of error is sufficiently small, credibility endogenously adjusts in such a way that makes preemption costless. Meanwhile, observational learning can cause existing errors to propagate through the market. This is because, like consumers, firms cannot detect faking. Thus, an erroneous report by one firm will make other firms more confident that the story is true, and thus more inclined to fake. The effect of observational learning is especially salient when there is a preemptive motive: in such cases, greater confidence in the story makes the threat of preemption more imminent, further exacerbating faking.

Reporting dynamics take two different forms in equilibrium: gradual changes in the absence of new reports and discrete changes in response to a new report. In the absence of new reports, firms gradually become more truthful, i.e., less inclined to fake. Furthermore, whenever there is a preemptive motive, firms become gradually more credible in the eyes of consumers. In other words, consumers are skeptical of quick reports, a finding which conforms with documented concerns about hasty news reporting. The reason for this gradual improvement in credibility lies in the firms incentives. The risk of preemption introduces an endogenous cost to delay. The firm must somehow be compensated for this cost to ensure that its indifference condition is satisfied. This is achieved by means of increasing credibility. That is, increasing credibility mitigates the haste-inducing effects of preemption enough to yield the firm indifferent between faking and truth telling.

Meanwhile, a report by a rival firm will cause a discrete change in a firm's reporting behavior. This can take the form of a *copycat effect*, in which one firm's report causes an instantaneous and persistent boost in faking by others. I show that the copycat effect is always intensified by observational learning, and is thus the channel through which observational learning propagates errors. When the copycat effect occurs, firms are not only herding on the decision to report, but also the timing of their reports. Furthermore, this herding on report timing applies not only to errors, but valid stories as well. In addition to anecdotal evidence of clustering in the timing of news errors [1], such herding

[1] Examples include the reporting errors surrounding the Boston bombings and the 2000 US presidential

has been documented in the empirical literature. Notably, Cagé, Hervé, and Viaud (2020) find that in 25% of cases, a news story is reported by a different media outlet within 4 minutes of being published by the original news breaker. I thus provide rationale for such herding that is grounded in both the strategic and learning environment news media face.

Finally, this paper also sheds light on the consequences of media mergers. Anderson and McLaren (2012) argue that media mergers could have consequences for consumers beyond those in conventional markets, namely that they can worsen media bias. I demonstrate that mergers can also impact news accuracy, but a subtle way. Specifically, a merger can mitigate faking early on in a news cycle by eliminating preemptive motives, but can exacerbate faking later as the effects of market consolidation take over. Thus, in contrast to Anderson and McLaren (2012), I argue that media mergers may under certain circumstances positively impact news quality.

Related Literature This paper adds to the literature on preemption in games. Originally studying optimal technology adoption (Fudenberg, Gilbert, Stiglitz, and Tirole (1983), Fudenberg et al. (1983)), such games model scenarios where players face some exogenous benefit from delaying action, but are rewarded for acting before their rivals. Broadly, I contribute to this literature by assuming there is no exogenous benefit to delay. Rather, the effect of timing and preemption on payoffs is dictated by a credibility function, which is endogenous. Despite this, even if no such benefit exists exogenously, I find that it arises in equilibrium whenever there is a preemptive motive. Furthermore, I show that firms may be endogenously rewarded for succeeding their rivals, thus nullifying any preemptive motive that firms face. In other words, I show that even if there is an exogenous benefit from preemption, it can be completely mitigated in equilibrium.

I contribute more specifically to the literature on observational learning in preemption games. In Hopenhayn and Squintani (2011) and Bobtcheff, Levy, and Mariotti (2022), players learn about their opponents' propensity to act by observing how long they have lasted without having done so. I instead consider a setting where players learn observationally about a variable of common value, i.e., where there are learning externalities. Such has been studied by Moscarini and Squintani (2010), Bobtcheff et al. (2022), Chen, Ishida, and Mukherjee (2023). Both Moscarini and Squintani (2010) and Bobtcheff et al. (2022) study winner-takes-all research races, with good and bad news learning, respectively. Despite considering a winner-takes-all setting, Moscarini and Squintani (2010) document herding in the timing of actions, wherein players quit the

election.

race simultaneously. Such behavior is also documented by Chen et al. (2023), who study a market entry game that is not winner-takes-all. The herding I document is different in nature: a report by one firm does not cause their opponent to immediately and deterministically follow suit, but rather induces a discrete and persistent rise in the probability that others act.

Thus, this paper connects to the literature on herding with endogenously-timed decisions without payoff externalities (Chamley and Gale (1994), Grenadier (1999), Murto and Välimäki (2011)). Chamley and Gale (1994) and Grenadier (1999) model investment timing games, showing that endogenously-timed information cascades, where one player's action triggers others to immediately follow suit, can take place. Meanwhile, Murto and Välimäki (2011) document dynamics and herding that are qualitatively similar to the equilibrium of this paper. Namely, players exit the game with a time-varying hazard rate that rises when an opponent exits. That is, herding is not deterministic but probabilistic. While mixing occurs in their setting because information cascades are inconsistent with equilibrium, in my setting mixing is driven by the endogeneity of the payoff function. Further, herding in my setting is not due purely to observational learning, but also firms' preemptive motive.

In application, this paper contributes to a recent literature on preemption in news (Lin (2014), Pant and Trombetta (2019), Andreottola, de Moragas, et al. (2020)). As in this paper, Lin (2014) considers a setting where firms dynamically learn about a story and must decide whether and when to report it. I contribute to this literature by modeling two important elements of the breaking news setting: credibility and observational learning. These features are significant because they drive the qualitative features of equilibrium, including reporting dynamics and herding. More broadly, this paper contributes to a literature on competition in news, as surveyed by Gentzkow and Shapiro (2008). A subset of this literature (Chen and Suen (2019) and Galperti and Trevino (2020)) consider the effects of competition on news accuracy when firms face costs or constraints to news accuracy. In contrast, I consider a setting where accuracy is not intrinsically costly but rather entails a strategic cost, namely that of being preempted. I contribute more generally to this literature by studying the effects of competition on not only news quality as a whole, but its dynamics. Namely, I show that competiton can give rise to dynamics in reporting behavior that are otherwise absent.

Finally, the notion of faking shares common threads with other work. Boleslavsky and Taylor (2020) study faking in a competition-free setting that encorporates discounting. Furthermore, the endogenous Poisson arrival of inaccurate information also arises in Che and Hörner (2018), and consists of spamming by recommender systems.

The remainder of the paper is organized as follows. Section 2 presents the model. In Section 3, I characterize the equilibrium. In Section 4, I present the core economic implications of this equilibrium, which pertain to the effects of competition and dynamics. In Section 5, I discuss media mergers. Finally, I present comparative statics and an extension where firms have heterogeneous learning abilities in sections 6 and 7, respectively. Section 8 concludes. All formal proofs are relegated to the Appendix.

2. A model of breaking news

There are $N \geq 1$ firms, indexed by i, and one consumer. Time, which is continuous and has an infinite horizon, is denote by $t \in [0, \infty)$. There exists some story, and the time-invariant state $\theta \in \{0, 1\}$ denotes whether it is true ($\theta = 1$) or false ($\theta = 0$). At $t = 0$, all players are endowed with a common prior $p_0 \equiv Pr(\theta = 1) \in (0, 1)$.

Learning and reporting Starting from $t = 0$, firms learn about the state. They do so by means of one-sided Poisson signals: if $\theta = 1$, a private signal revealing that $\theta = 1$ arrives to each firm at a Poisson rate $\lambda > 0$, where the time of this arrival is independent across firms. Formally, letting $s_i \in [0, \infty]$ denote the time at which such a conclusive signal arrives to firm i, with $s_i = \infty$ denoting that a signal never arrives, $s_i \sim (1 - e^{-\lambda s_i})$ if $\theta = 1$, and $s_i = \infty$ if $\theta = 0$. This learning process is meant to approximate how firms conduct research on breaking news: rather than seeking piecemeal evidence, they pursue reliable sources who can confirm the story. In the case of a terrorist attack, this could entail reaching out to contacts at the police department.

Firms choose whether and when to report the story. Specifically, at any point in the game, a firm can choose to make a report as long as they have not already done so. As the payoff function will soon illustrate, the content of this report can be interpreted as an assertion that the story is true, i.e., that $\theta = 1$. A report history H is a partially ordered set of pairs (i, t_i), pairing each firm i who has reported with a report time t_i, with elements ordered according to the order in which the reports were made.[2] Report histories are public: all players observe the current report history.

Payoffs A firm who never reports earns a payoff of 0. A firm who does report earns

$$k_n \alpha - \beta \mathbb{I}[\theta = 0]. \tag{1}$$

[2] Elements are ordered according to relation $\succsim$, where $(i, t_i) \succ (j, t_j)$ if $t_i > t_j$ or $t_i = t_j$ but i reported first, and $(i, t_i) \sim (j, t_j)$ if the reports were made simultaneously.

The first term ($k_n\alpha$) is the market share (i.e., viewership or readership) that the firm enjoys from reporting a story. It is the product of k_n, a parameter capturing the role of the firm's order n, and α, the credibility of the report. More precisely, the index n denotes that the firm was the n^{th} to report, i.e. $n = |H| + 1$, where H denotes the current history at the time of the report. I assume that $k_1 \geq k_2 \geq ... \geq k_N \geq 0$, i.e., firms who report earlier than their competitors earn greater market share, all else equal. Meanwhile, credibility α denotes the consumer's belief, at the time that the report is made, that the firm has received independent confirmation of the state. Formally, it is the belief that $s_i \in [0, t]$, where t is the time of the report. In assuming a product form for market share, I take the stance that consumers value accuracy in journalism, and thus only consume news to the extent that they find it credible. Furthermore, this formula for market share can be microfounded by modeling a continuum of consumers with a preference for accuracy who multi-home across news firms. While I do not formally present this microfoundation in the main text, I do so in Appendix A. The second term of (1), $-\beta\mathbb{I}[\theta = 0]$, is the penalty of error: a firm who reports when $\theta = 0$ incurs a penalty $\beta > 0$. This captures the reputational harm a firm suffers from making a report that is later uncovered to be false.

Equilibrium A Markov strategy F is a set of distributions $F_{p,n}$ over future report times for each belief $p \equiv Pr(\theta = 1)$ and order n of the next firm to report. [3] Specifically, the span of time the firm waits before reporting, conditional on not receiving a conclusive signal, is distributed according to $F_{p,n} \in \Delta[0, \infty]$ where ∞ denotes a lack of report altogether.[4] I restrict attention to symmetric equilibria, and thus omit the firm's index in much of the analysis below.

I place some restrictions on F. First, I assume that for all (p, n), $F_{p,n}$ must be piecewise twice differentiable and right-differentiable everywhere on $[0, \infty)$. This grants analytical convenience and ensures that equilibrium objects are well-defined.[5] Second, I impose a selection criterion (SC): a firm immediately reports a story it knows is true. This is stated as Definition 1.

[3] I assume that if multiple firms report at the same history H, one firm will be assigned order n, another $n + 1$, etc., with their identities randomly determined according to a uniform distribution.

[4] By defining strategies in this way, firms can react instantly to a competitor's report. For instance, if $F_{p,2}(t) = 1$ for all t and p, then if some firm makes the first report at t, all other firms will also report at t.

[5] Note that F satisfies the above restrictions if and only if there exist two functions on p, q_n and b_n, where for all (p, n) and $t \geq 0$,

$$F_{p,n}(t) = \sum_{s \leq t | q_n(p(s)) > 0} q_n(p(s)) + \int_0^t b_n(p(s))ds$$

such that b_n is piecewise differentiable and $q_n(p) = 0$ at all but a countable number of p. Namely, q_n denotes the *point mass* of reports, while b_n denotes the right *arrival rate* of reports.

Definition 1. F satisfies (SC) if

$$F_{1,n}(0) = 1 \text{ for all } n \in \{1, ..., N\}.$$

This criterion rules out equilibria with periods of silence supported by pessimistic off-path beliefs, i.e., beliefs that reports made during these gaps have little or no credibility. Furthermore, (SC) implies that fixing an n and starting belief p, all remaining players hold the same common belief about the state after t time has passed, assuming no new reports are made. This common belief is denoted by $p(t)$, and it follows from Bayes Rule that:

$$p(s) = \frac{pe^{-\lambda(N-n+1)s}}{pe^{-\lambda(N-n+1)s} + (1-p)}. \tag{2}$$

Defining strategies in this way, i.e. with a separate distribution for each (p, n), is convenient but introduces redudancy. Namely, for any (p, n) and $t > 0$, $F_{p,n}$ and $F_{p(t),n}$ "overlap": both distributions specify the firm's reporting behavior at $(p(t+s), n)$ for any $s \geq 0$. Thus, I impose that the $F_{p,n}$ must be mutually consistent.[6] That is, at any (p, n) on-path and $t > 0$,

$$F_{p(t),n}(s) = \frac{F_{p,n}(t+s) - F_{p,n}(t_-)}{1 - F_{p,n}(t_-)} \text{ for all } s \geq 0 \text{ whenever } F_{p,n}(t) < 1, \tag{3}$$

where $F_{p,n}(t_-) \equiv \lim_{\tau \uparrow t} F_{p,n}(\tau)$. This formula is a result of Bayes Rule. In what follows, let $\mathcal{F}$ denote set of distributions F that satisfy the above restrictions.

Before proceeding, I define two terms to describe reporting: faking and truth telling. A report is fake if it is made despite the firm lacking independent confirmation, i.e., a signal $s^i \neq \emptyset$. Meanwhile, a report that is made after the firm has confirmation is *truthful*. Note that under the selection assumption (SC), strategies differ only in their distributions over fake reports.

I seek a symmetric Markov perfect equilibrium of this game. This is a Markov strategy F paired with beliefs α and p at each history such that F is sequentially rational and the beliefs are consistent with Bayes Rule. The consistency of α with Bayes Rule implies the following at all (p, n) on-path: [7]

[6] This condition is analogous to the closed-loop property specified in Fudenberg and Tirole (1985). I adopt the term *consistency condition* from Laraki, Solan, and Vieille (2005), who define this condition for a general class of continuous-time games of timing.

[7] The formula is derived by applying Bayes Rule to a discrete-time approximation of the beliefs that obtain under this game. This derivation is presented in Appendix B.

$$\alpha_n(p) = \begin{cases} \frac{\lambda p}{\lambda p + b_n(p)} & \text{if } F_{p,n}(0) = 0 \\ 0 & \text{if } F_{p,n}(0) > 0 \end{cases} \quad (4)$$

where $b_n(p) \equiv F'_{p,n}(0+)$, the right-derivative of $F_{p,n}$ at 0, is the instantaneous arrival rate of fake reports.

This formula is intuitive. If $F_{p,n}(0) > 0$, there is a point mass of fake reports at (p, n). Meanwhile, because conclusive signals are distributed continuously over time, the instantaneous probability of a truthful report is zero. So, the consumer and all competing firms are certain that a report made at (p, n) was fake, and thus assign to it zero credibility. If there is not a point mass of fake reports at (p, n), credibility is assessed by comparing the arrival rates of truthful reports (λp) to that of fake reports ($b_n(p)$), assigning higher credibility to reports made when the arrival rate of fake reports is relatively low.

3. Equilibrium characterization

This section presents the equilibrium characterization. I first establish two properties that are instrumental to the analysis. Then, as a stepping stone to the full model characterization, I consider the monopoly case. This elucidates the forces at play even when competition is absent. In particular, I show that errors may occur even without competition, and that such errors are driven by a lack of commitment power by the firm. Finally, I characterize the equilibrium of the full model, establishing existence and uniqueness.

3.1. The firm's problem

Here, I present the firm's problem. I begin by defining a useful object, the *first report distribution*. Fix a report history H and strategy profile F, and let p denote the common belief and n the order of the next firm to report. Index the firms who have not yet reported by i. The first report distribution $\Psi^i(s)$ denotes the probability that player i reported when or before s time has passed and was not preempted by any of the remaining firms (i.e., i was the first to make a new report).[8] This is given by:

$$\Psi^i(s) = p\int_0^s e^{-\lambda r(N-n)} \prod_{j\neq i}(1-F^j_{p,n}(r))d(e^{-\lambda r}(F^i_{p,n}(r)-1)) + (1-p)\int_0^s \prod_{j\neq i}(1-F^j_{p,n}(r))dF^i_{p,n}(r).$$

The firm's value from playing strategy F^i at (p, n) given each of its opponents plays F^j

[8] While Ψ is a function of F, p, and n, I omit these dependences for brevity.

can then be written recursively as

$$V_{p,n}(F^i) = \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s) + \sum_{j \neq i} \int_0^\infty V_{p^j(s),n+1}(F^i) d\Psi^j(s), \tag{5}$$

where $V_{p,N+1} \equiv 0$ and $p^j(s)$ denotes the common belief when s time has passed, conditional on no new reports having been made, except for a report by j at time s. The first integral of (5) is firm i's expected payoff from reporting conditional on being the first of the remaining firms to do so, and the second integral is conditional on being preempted. Specifically, upon being preempted by j at time s, the state changes discretely from $(p(s), n)$ to $(p^j(s), n+1)$ (i.e., both the common belief and the order of the next firm changes). Thus, the firm's continuation value upon being preempted is its value at this new state.

The firm's problem at (p, n) is then given by

$$\max_{F^i \in \mathcal{F}} V_{p,n}(F^i).$$

3.2. Properties of equilibrium

I begin by presenting two necessary conditions on a firm's equilibrium strategy that are instrumental to the analysis. First, I show that there cannot exist any jumps (i.e., point masses) in the distribution of fake reports. Second, whenever a firm is less-than-fully credible, it must satisfy certain indifference conditions. Similar properties arise in other games with continuous strategy spaces, where they result from competition.[9] However, as I will illustrate below, here they are instead driven by the endogenous nature of credibility and thus hold even without competition.

First, let us consider the "no jumps" property, which is formalized as Lemma 1:

Lemma 1. *In equilibrium, at any (p, n) on-path, $F_{p,n}$ is continuous everywhere whenever $p < 1$.*

This states that fake reports are distributed continuously over time whenever a firm is not certain that the story is true. I.e., there can never be a point mass in faking when $p < 1$. To see why, recall that a report made when there is a point mass of faking yields zero credibility. Meanwhile, faking while also not being certain than the story is true yields a strictly positive expected penalty $\beta(1 - p)$. Thus, a firm's value from faking at such a time is strictly negative. The firm could then profitably deviate by truth telling: this would preclude the firm from making an error, ensuring a weakly positive payoff.

[9] These include war of attrition games (Hendricks, Weiss, and Wilson (1988)) and all-pay auctions (Baye, Kovenock, and De Vries (1996)).

Now let us state the indifference property. To this end, let δ_s for $s \in [0, \infty]$ denote the distribution that places full mass on faking after s time has passed. In particular, δ_0 denotes immediate faking, while δ_∞ denotes that the firm never fakes (i.e., is truthful).

Lemma 2. *In equilibrium if $\alpha_n(p) < 1$ and (p, n) is on-path, then there exists an $\varepsilon > 0$ such that*

$$V_{p,n} = V_{p,n}(\delta_s) \text{ for all } s \in [0, \varepsilon) \cup \infty,$$

where $V_{p,n}(\delta_s)$ is the value from playing δ_s at (p, n) and F at (q, m) for all q and $m > n$.

Lemma 2 states that whenever $\alpha_n(p) < 1$, the firm must find a number of strategies optimal. First, it must be optimal to fake immediately (i.e., play δ_0). Second, it must be optimal to be truthful for some sufficiently short span of time dt and then fake (i.e., play δ_{dt}). Third, it must be optimal to never fake (i.e., play δ_∞). I will now provide some insight into the proof of this lemma, which is presented formally in the appendix. Let us begin by considering why δ_{dt} must be optimal for $dt \in [0, \varepsilon]$. It follows from our regularity conditions on the firm's strategy that $\alpha_n(p(s))$ must be right-continuous in s. This means that if $\alpha_n(p) < 1$, $\alpha_n(p(s)) < 1$ for all s sufficiently small. Furthermore, we recall that whenever $\alpha_n(p(s)) \in (0, 1)$, the firm is faking with a strictly positive hazard rate. This means that the firm mixes between faking with delay $[0, \varepsilon]$, implying that all such pure strategies are optimal. Next, let us consider why never faking (playing δ_∞) must be optimal. Suppose by contradiction that it is not. Then, a firm who has not received a conclusive signal must fake with probability 1. To achieve this, the firm must sustain a sufficiently high hazard rate of faking as t tends to ∞. But because the hazard rate of truthful reports tends to zero as p shrinks, credibility must tend to zero, making faking suboptimal.

3.3. The monopoly characterization and the role of commitment

I now characterize the equilibrium under a monopoly, i.e., assuming $N = 1$. As there is only one firm, I drop the n index from all functions and parameters.

Claim 1. *Under a monopoly, for all p on-path*

$$\alpha(p) = \min\{\beta/k, 1\}.$$

Qualitatively, Claim 1 states that the monopolist's credibility is constant over time and not always perfect. In particular, credibility is weakly increasing in β/k, and less-than-perfect whenever $\beta/k < 1$. That is, errors can occur when the ex-post penalty of error is relatively low. In the remainder of this subsection, I provide intuition for these properties and show that the monopolist's errors are driven by their inability to commit to a reporting strategy.

Let us first consider why credibility is constant. Recall from Lemma 2 that whenever $\alpha_n(p) < 1$, the firm must be indifferent between faking immediately and after some short wait dt. By the martingale property of firm's belief p, both of these strategies yield the same expected penalty from error $\beta(1-p)$. So, for both strategies to be optimal, they must also yield the same expected prize $k\alpha$. Thus, credibility must remain constant. It is noteworthy that this reasoning is predicated on the fact that waiting is costless. Indeed, this is true under a monopoly. Not only is waiting intrinsically costless (i.e., future payoffs are not discounted), a monopolist does not incur the strategic cost to waiting that preemption might entail. As I show in Section 4, this strategic cost of waiting is precisely what gives rises to dynamics in credibility when there are multiple firms in the market.

Though a monopolist's credibility is constant, its strategy is dynamic: a firm who fakes will become gradually more truthful over time. Specifically, the hazard rate of faking (b) strictly decreases and tends to zero whenever credibility is less than one. This follows from (4), and is due to the fact that as more time passes without a report, the common belief drifts down. This is an artifact of the firm's one-sided Poisson learning process: the absence of a report means that the firm has not received a conclusive signal, an event that is consistent with $\theta = 0$. So for credibility to remain constant, the hazard rate of faking must decline and tend to zero.

Now, let us consider why truth-telling cannot be sustained when $\beta/k < 1$, and why credibility is equal to β/k. Suppose by contradiction that $\beta/k < 1$ and the firm is truthful. This implies full credibility, and thus that the market share ($k\alpha(p)$) exceeds the penalty of error (β). So, it is strictly optimal to report, even if the story is false. That is, the firm can profitably deviate by faking. We conclude that in any equilibrium, the firm fakes with positive probability. To pin down the level of credibility, we recall from Lemma 2 that a firm who fakes must be indifferent between faking immediately and remaining truthful. Indeed, there is a unique value of credibility that ensures this indifference: β/k. There is some intuition behind this: the bigger β/k is, the more costly errors are compared to market share for any α. Resultedly, the more costly faking is to truth-telling. So as β increases, α must correspondly increase to maintain indifference.

In this model, I assume that consumers cannot detect faking and firms cannot commit to a reporting strategy. Rather, a firm optimizes its strategy, for instance by faking, taking the credibility function as given. This can tempt news firms to capitilize on their credibility by faking, thus causing a deterioration in equilibrium credibility. I show that taking away this temptation, i.e. allowing the monopolist to commit, errors would never take place.

To show this, I consider a modified version of the model where the firm announces, and commits to, a reporting strategy before the consumer assesses credibility, presented formally in Appendix D. Claim 2 states that in this setting, a monopolist never fakes, and thus never errs.

Claim 2. *Under commitment, the unique monopolist equilibrium is one in which $b(p) = 0$ for all p on-path.*

One can immediately see that given the ability to commit, the firm would always choose truth telling over its non-commitment equilibrium strategy, even when $\beta < k$. By committing to truth telling, the firm is guaranteed a payoff of k if $\theta = 1$, and 0 if $\theta = 0$. Meanwhile, under the no-commitment equilibrium, the firm will earn strictly less when $\theta = 1$, due to its strictly lower credibility, and earn 0 when $\theta = 0$: in the model without commitment, the firm's payoff from faking exactly offsets the penalty of error. Intuitively, committing to truth-telling is better for the firm because the enhanced credibility the firm enjoys when the story is true exceeds any payoff it might enjoy from reporting a false story (which is zero in equilibrium). Indeed, truth telling is not only superior to the non-commitment solution, it is the unique commitment solution. This illustrates an important point about faking: it not only deteriorates the quality of information consumers receive, it also harms firms. Despite this, faking happens because firms cannot credibly promise truthfulness to consumers.

3.4. Full model characterization

Now, I characterize the equilibrium of the full model (i.e., under an arbitrary N), establishing both existence and uniqueness. I show that any equilibrium is the solution to a recursive set of boundary value problems. Specifically, whenever the firm is not truthful, credibility must satisfy an ODE and boundary condition.

Let us begin by deriving the conditions under which the firm is truthful. This both serves as a stepping stone to a full characterization and illustrates how competition can deteriorate credibility and exacerbate faking. This result is stated as Proposition 1.

Proposition 1. *In equilibrium, at any (p, n) on-path, $\alpha_n(p) = 1$ if and only if:*

1. $k_n \leq \beta$
2. $p \leq p_n^* \equiv \min\{\frac{k_n - \beta}{\frac{k_n}{N-n+1} - \beta}, 1\}$.

Proposition 1 provides two conditions that are necessary and sufficient for truth telling. Recall that the first condition alone, $k_n \leq \beta$, was sufficient for truth telling under a monopoly (Claim 1). However, under competition, a second condition is required: the common belief must lie below some threshold p_n^*.

The need for this additional condition illustrates an important point: truth telling is harder to sustain under competition. Under a monopoly, the only cost to truth telling is that the firm will fail to earn any market share when $\theta = 0$. But under competition, truth telling also entails a risk of being preempted. Namely, a truthful firm risks its opponents reporting first, either because they have learned the story is true or because they are faking. Assuming that being preempted is costly, one can see why truth telling is harder to sustain.

But in this model, we cannot take for granted that being preempted is costly. It is, however, true that preemption is costly conditional on being truthful in equilibrium. This is most obvious in a *winner takes all* setting, where $k_n = 0$ for all $n > 1$. That is, all firms with the exception of the first to report earn zero market share. In this case, the costliness of being preempted is an artifact of the parameters, as a preempted firm can earn at best zero payoff. In general, the decreasing nature of k_n alone does not imply that preemption is costly: improved credibility for succeeding firms could endogenously counteract the decay in k_n, making preemption costless or even valuable. Indeed, I will show in the next section that under certain parameters, precisely such a phenomenon occurs in equilibrium. But conditional on a firm being truthful in equilibrium, this cannot happen: truthfulness implies full credibility, leaving no room for a succeeding firm to improve on it.

Let us now consider why under competition, truthtelling is only possible when firms are sufficiently pessimistic about the story. This can be explained by the fact that faking and truth telling each pose a different kind of risk to the firm: while truth telling entails the risk of being preempted, faking entails the risk of making an error and incurring penalty β. Both of these depend on the belief p about the state: higher p implies a lower probability of error but also a higher probability of being preempted. The former is immediate, and the latter is due to the fact that preemption is more likely when the story is true. Namely, conditional on the story being true, an opponent reports not just because it is faking, but also because it has received confirmation. Thus, a firm with a higher p will believe its risk of being preempted is higher, too. Because a lower risk of error and higher risk of preemption make faking relatively more profitable, truth telling is harder to sustain when p is high.

While Proposition 1 pins down the conditions under which the firm is truthful, it remains to characterize the firm's behavior when truth telling does not hold. To this end, I obtain a key result: when the firm fakes, credibility must satisfy an ODE and limit condition.

Proposition 2. *In equilibrium, at all (p, n) on-path where $k_n \geq \beta$ or $p > p_n^* \equiv \frac{k_n - \beta}{k_n/n - \beta}$, the* *following ODE must be satisfied:*

$$\alpha_n'(p) = -\frac{1}{k_n(1-p)\alpha_n(p)}\frac{N-n}{N-n+1}[k_n\alpha_n(p) - V_{\tilde{p},n+1} - \beta(1-\alpha_n(p))(1-p)], \qquad \text{(ODE)}$$

where $\tilde{p} \equiv \alpha_n(p) + (1-\alpha_n(p))p$*.*

In addition, $\lim_{p\to 0+}\alpha_n(p) = \beta/k_n$ *must hold if* $k_n > \beta$ *and* $\lim_{p\to p_n^*+}\alpha_n(p) = 1$ *if* $k_n \leq \beta$*.*

The proof for Proposition 2 follows from the indifference condition established in Lemma 2. Namely, when credibility is less than 1, there exists an $\varepsilon > 0$ such that the strategies δ_Δ yield the same payoff for all $\Delta \in (0, \varepsilon]$. This implies

$$\frac{d}{d\Delta+}V_{p,n}(\delta_\Delta)\bigg|_{\Delta=0} = 0. \qquad (6)$$

It follows from (5) that

$$V_{p,n}(\delta_\Delta) = \int_0^\Delta k_n\alpha_n(p(s))d\Psi^i(s) + (N-n)\int_0^\Delta V_{p^{-i}(s),n+1}d\Psi^{-i}(s) + \\ (1 - \sum_j \lim_{s\to\Delta-}\Psi^j(s))[k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta))],$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play the equilibrium strategy $F_{p,n}$. Differentiating, we obtain

$$\lim_{\Delta\to 0+}\frac{d}{d\Delta}V_{p,n}(\delta_\Delta) = [\frac{dp}{dt}(k_n\alpha_n'(p)) - \frac{\lambda p(N-n)}{\alpha_n(p)}(V_{\tilde{p},n} - V_{\tilde{p},n+1})]. \qquad (7)$$

Setting the right-hand side to zero, in accordance with (6), yields (ODE). Equation (7) is intuitive. Waiting to fake, rather than faking immediately, has two consequences for the firm's payoff. The first is that α_n, and thus the prize from reporting, may change. This rate of change in credibility is $\frac{dp}{dt}(k_n\alpha_n'(p))$. The second consequence is that the firm risks being preempted: this happens at a Poisson rate $\frac{\lambda p(N-n)}{\alpha_n(p)}$, in which case its expected payoff changes by $V_{\tilde{p},n} - V_{\tilde{p},n+1}$. I call this decrease in value the firm's *regret* from preemption.

Let us examine the rate and regret of preemption more closely. As one might expect, the rate of preemption is increasing in the number of rival firms remaining ($N-n$) and the expected rate at which these rivals can confirm the story (λp). It is also decreasing in credibility: less credible firms are more likely to fake, and thus more likely to preempt. Meanwhile, the regret of preemption is the difference between two values, $V_{\tilde{p},n+1}$ and $V_{\tilde{p},n}$. $V_{\tilde{p},n+1}$ denotes the firm's continuation value in the event that it is preempted at (p,n). This value is taken at $(\tilde{p}, n+1)$ because preemption affects both the firms order and the common belief. Namely, while the common belief was p prior to the rival firm's report, it increases to $\tilde{p} \equiv \alpha_n(p) + (1 - \alpha_n(p))p$ in the immediate aftermath of the report. This expression for $\tilde{p}$ demonstrates that a rival firm's report means one of two things: either the report was triggered by a conclusive signal, in which case the new belief should be 1, or it was fake, in which case the new report offers no new information and the belief remains p. Since faking is unobservable, the new common belief $\tilde{p}$ is an average of these two conditional beliefs, where the weight given to the report being informed is its credibility. Meanwhile, $V_{\tilde{p},n}$ denotes the continuation value conditional on not being preempted. Notably, this value is not assessed at the belief prior to preemption p, but rather the posterior $\tilde{p}$. In this sense, $V_{\tilde{p},n+1} - V_{\tilde{p},n}$ denotes firm's regret from not having reported after being preempted.

In addition to (ODE), Proposition 2 establishes that one of two limit conditions must hold. Which condition holds depends on the model parameters, and like (ODE), these conditions result from the firm's indifference condition. First consider the case where $k_n \leq \beta$. Recall from Proposition 1 that in this case, $\alpha_n(p) = 1$ whenever $p \leq p_n^*$. It follows that $\alpha_n(p)$ limits to 1 as the belief approaches p_n^*. If it did not, then credibility would exhibit an upward discontinuity at p_n^*. This means that at beliefs close to p_n^*, the firm could profitably deviate by waiting until p_n^* is reached to fake. Thus, the indifference condition would fail. When $k_n > \beta$, the firm never truth tells in equilibrium, and thus the indifference condition must always be satisifed. As the common belief p approaches zero, a firm who fakes does so being nearly certain that its report is erroneous, and will incur penalty β. Thus, the firm's payoff from faking limits to the following:

$$\lim_{p \to 0+} V_{p,n}(\delta_0) = k_n \lim_{p \to 0+} \alpha_n(p) - \beta.$$

Lemma 2 also states that the firm must find truth telling (δ_∞) optimal. As $p \to 0+$, the value of truth telling tends to zero, as it becomes increasingly likely that the firm never reports. The limit condition in this case, $\lim_{p \to 0+} \alpha_n(p) = \beta / k_n$, is precisely what is needed to ensure indifference between faking and truth telling.

To take stock, Proposition 1 and Proposition 2 provide two necessary conditions on equilibrium credibility. They pin down the region in which truth telling occurs (Proposition 1), and show that otherwise, credibility must satisfy a recursive boundary value problem (Proposition 2). One can show that these two conditions are sufficient for an equilibrium as well, provided that the firms strategy is consistent with this credibility function.[10] To prove this, one must show that if credibility statisfies these conditions, the firm cannot profitably deviate from the strategy that is consistent with this credibility. On

[10] This result is presented in the Appendix as Lemma 5.

the region where credibilty is perfect, a deviation would consist of faking. Proposition 1 establishes that such a strategy cannot be played in equilibrium, that is, the firm could profitably deviate by truth telling even when their opponents are faking (i.e., the risk of being preempted is higher) and credibility is less-than-perfect (i.e., the benefit of reporting is lower). Such a strategy thus cannot be more profitable than truth telling when the firm's opponents are not faking, and credibility is perfect. On the region where $\alpha_n(p) < 1$, the firm's strategy involves mixing between faking and truth telling. This too must be optimal, because both (ODE) and the boundary conditions guarantee it. In particular, a credibility function satisfies these conditions if and only if it implies indifference between faking and truth telling.

Thus, the equilibrium is fully characterized by the solution to a recursive set of boundary value problems. While I do not have a closed-form solution to this problem, I use the Picard-Lindelof theorem to establish existence and uniqueness. This result is stated as Theorem 1.

Theorem 1. *There is a unique equilibrium, where uniqueness applies at* (p, n) *on-path.*

4. Dynamics and herding

With the above characterization in hand, I study dynamics. I show that credibility gradually improves over time whenever preemption is costly, with discrete changes in reporting behavior triggered by the report of a rival firm. Under certain conditions, in particular when observational learning is sufficiently strong, firms herd on their opponents' decisions to report as well as the timing of these reports. This is due to a *copycat effect*, wherein one report causes a surge in faking by others.

The nature of these dynamics will hinge on whether the last firm fakes in equilibrium. Thus, I will discuss two separate cases in turn: the first where the last firm is truthful ($k_N \leq \beta$) and the second where the last firm fakes with positive probability ($k_N > \beta$). I show that firms face a preemptive motive when the last firm is truthful, but this motive endogenously disappears otherwise. I begin by showing that credibility strictly improves over time when $\beta > k_N$, as long as no new reports are made.

Proposition 3. *If* $\beta > k_N$*,* $\frac{d}{dt}\alpha_n(p(t)) > 0$ *and* $\frac{d}{dt}b_n(p(t)) < 0$ *whenever* $\alpha_n(p(t)) < 1$*, for all* (p, n) *on-path.*

The broad implication of this result is that while credibility is constant under a monopoly, competition can give rise to dynamics. To understand why, it can be helpful to observe the following: as long as $\alpha_n(p(t))$ has not reached its upper bound of 1, it must strictly increase precisely when there is a positive regret to preemption. Formally, this follows from (ODE). It is especially clear when we write (ODE) in the following form:

$$\frac{d}{dt}\alpha_n(p(t)) = \frac{\lambda p(N-n)}{\alpha_n(p(t))k_n}[V_{\tilde{p},n} - V_{\tilde{p},n+1}]. \tag{8}$$

We can see that $\alpha_n(p(t))$ is strictly increasing if and only if the regret of preemption, $V_{\tilde{p},n} - V_{\tilde{p},n+1}$, is strictly positive. There is intuition behind this result. Whenever the firm is less-than-fully credible, it must be indifferent between faking immediately and waiting some length of time before doing so. However, if credibility remained constant, reporting immediately would be strictly better —it would allow the firm to avoid being preempted while suffering no harm to its credibility. To restore indifference, the firm must somehow be compensated for waiting. This can only be achieved by means of increasing credibility: while by waiting the firm risks being preempted, it will enjoy higher credibility otherwise. That is, credibility must increase to mitigate the haste-inducing effects of preemptive risk.

We have argued that credibility must increase when there is a positive regret from preemption. However, as discussed above, this is not necessarily true even when there are multiple firms in the market. But it is indeed true that preemption is costly when $\beta > k_N$, i.e. when β is high enough to ensure the last firm to report is truthful. The proof requires a backwards induction argument, but its core reasoning is most easily illustrated in a duopoly setting ($N = 2$) where $\beta \in (k_2, k_1)$. In this case, a firm fakes with a positive hazard rate as long as nobody has reported yet, but switches to truth telling as soon as their opponent makes a report. Proposition 3 asserts that the credibility of the first report α_1 must strictly increase over time. To see why, suppose instead that α_1 is constant, as in the monopoly case.[11] Since $k_1\alpha_1(p(t))$ must limit to β (Proposition 3), it follows that $k_1\alpha_1(p(t)) = \beta$ for all t. That is, the market share from reporting first is always β, no matter when the report is made. This implies a failure of the firm's indifference condition: the market share from reporting first is so high that faking is strictly optimal. Specifically, if the story is false both faking and truth telling yield 0 payoff, but if the story is true the firm is ensured a payoff of β by faking but by truth telling risks being preempted and only earning k. To restore indifference, the market share of the first firm must instead be strictly less than β and approach it from below. This restores indifference because it increases the value of truth-telling in two ways: (1) the lower market share from reporting first lowers

[11] The argument made here is purely illustrative; it does not rule out the possibility that $\alpha_1(p(t))$ is increasing in t, nor that the function is only locally non-increasing. For a formal treatment, please see the proof.

the cost of being preempted and (2) as argued above, increasing α provides an additional incentive to wait.

Proposition 3 also states that the hazard rate of faking, $b_n(p(t))$, is decreasing in t, an immediate corollary of the increasing nature of credibility. While this same result obtains in the monopoly case, the strictly increasing nature of credibility implies that $b_n(p(t))$ decays more quickly than under the monopoly equilibrium. I.e., the firm's preemptive motive also gives rise to more extreme dynamics in faking.

So far, we have restricted attention to the case where $k_N < \beta$. In fact, one can show that if this does not hold, preemption becomes costless in equilibrium and the dynamics in α_n disappear. I formalize this as Proposition 4.

Proposition 4. *If $k_N > \beta$, $\alpha_n(p) = \beta/k_n$ for all (p, n).*

This result states that when $k_N \geq \beta$, credibility is constant at a level where the market share $k_n\alpha_n(p)$ is not affected by the firm's order. That is, firms enjoy higher credibility from reporting after their opponents, which mitigates the decline in k_n in such a way that makes preemption costless.

To understand the reasoning for this claim, it is again helpful to consider the duopoly case, but this time assuming that $\beta < k_2 < k_1$. It follows from the monopoly characterization that the market share for the second reporter, $k_2\alpha_2(p)$, equals β no matter when that report is made. Now let us consider the first reporter. Again, the market share of the first reporter must limit to β. But in this case, it cannot limit to β from below. If it did, a firm could profitably deviate by not being truthful: being preempted would *benefit* the firm, as it would yield a higher market share β. Instead, the market share of the first firm, $k_1\alpha_1(p)$, must always equal β: this ensures that the firm incurs no loss in value by being preempted, and so its indifference condition is preserved.

While not immediately obvious, Proposition 3 and Proposition 4 imply that competition exacerbates faking. To formalize this, let $\bar{b}_n$ denote equilibrium faking under a monopoly ($N = 1$) with maximal market share k_n. This denotes equilibrium faking under a counterfactual where competition is absent. Corollary 1 states that faking is always higher in equilibrium than under the competition-free counterfactual.

Corollary 1. *For any (p, n), $b_n(p) \geq \bar{b}_n(p)$, where the inequality holds strictly whenever $b_n(p) > 0$ and $\beta \in (k_N, k_n)$.*

To see why this holds, note that Proposition 2 establishes that under competition, credibility limits to the value that obtains under a monopoly ($\min\{\beta/k_n, 1\}$) as $p \to 0$.

Meanwhile, Proposition 3 and Proposition 4 establish that credibility is decreasing in p. This implies that credibility under competition lies below the monopoly value for all p, and thus, faking must lie above the monopoly value.

Before proceeding, let us take stock of these results. Proposition 3 asserts that under certain conditions, news reports that are made with greater delay for research are more trustworthy to consumers. I.e., all else equal, consumers will have greater trust in a firm's journalistic standards when a report is not made quickly. In this sense, this model provides justification for consumer distrust of hasty reporting that originates from the firm's preemptive motive. Meanwhile, Proposition 4 establishes a notable feature of equilibrium: competition alone does not imply preemptive concerns. Even though reporting first yields more market share, all else equal, payoffs may endogenously adjust in such a way that makes preemption costless. Because the firm's continuation value is determined inductively, the existence of a preemptive motive hinges on the incentives of the last firm to report. Finally, the fact that equilbrium credibility limits from below to the value that obtains under a monopoly demonstrates that competition exacerbates faking.

Proposition 3 and Proposition 4 describe the dynamics of reporting conditional on no new reports being made. For a more complete picture of equilibrium dynamics, it is helpful to plot simulations of credibility and faking over the course of time. Figure 1 does this for the case when $k_N < \beta$. As per Proposition 3, credibility is continuously increasing and faking continuously decreasing as long as no new reports are made. However, new reports trigger discrete jumps in credibility and faking. Notably, as illustrated by these graphs, these jumps need not be monotonic. Dynamics are qualitatively different when $k_N \geq \beta$. This is illustrated by Figure 2, which plots a simulation in this case of the parameters. As per Proposition 4, credibility is flat, with new reports triggering exclusively upwards jumps. But despite this, faking exhibits a discrete upwards jump in the aftermath of a new report.

Both these simulations illustrate the copycat effect, in which one firm's report causes a surge in faking by others. I define it formally below.

Definition 2. A report at (p, n) exhibits the *copycat effect* if

$$b_{n+1}(\tilde{p}) - b_n(p) > 0.$$

Let us consider what forces are responsible for this effect. To this end, recall that a new report affects two changes to the state. First, it increments the order of the next firm to report from n to $n+1$. Second, firms learn observationally from the report, and thus the

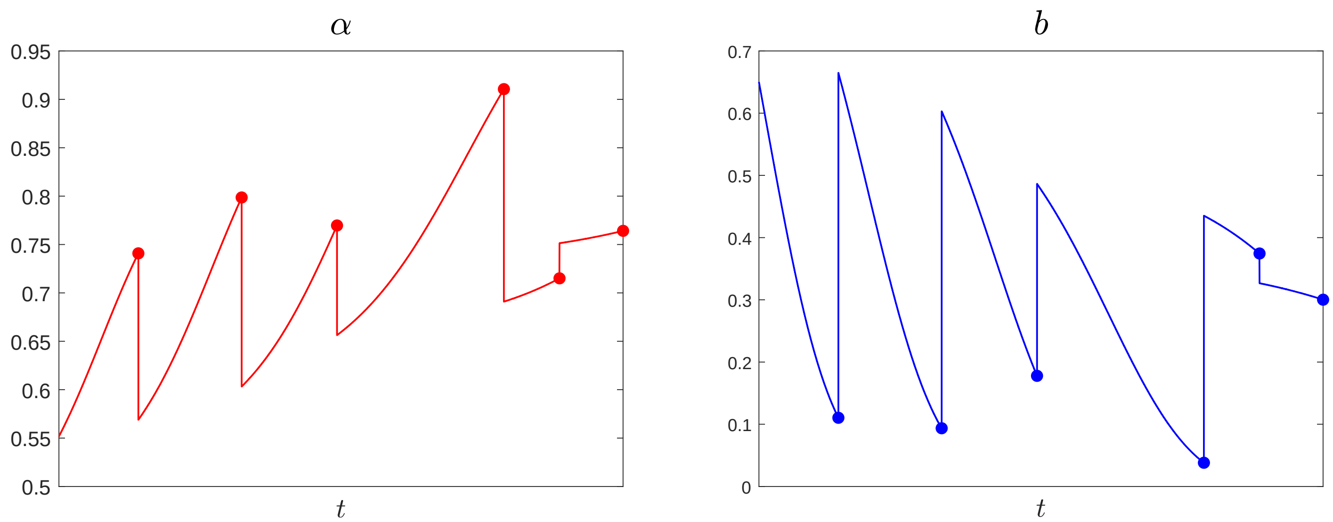

Figure 1: Simulation of crediblity (α) and the hazard rate of faking (b), over the course of a game when $k_N < \beta$. Discrete jumps signify that a firm has made a report. Upwards jumps in b illustrate the copycat effect.

common belief increases from p to $\tilde{p}$. The following decomposition isolates the respective impacts of these two changes:

$$b_{n+1}(\tilde{p}) - b_n(p) = \underbrace{[b_{n+1}(p) - b_n(p)]}_{\text{change in order}} + \underbrace{[b_{n+1}(\tilde{p}) - b_{n+1}(p)]}_{\text{change in belief}}.$$

In equilibrium, the change in order has an ambiguous effect on faking, i.e., $b_{n+1}(p) - b_n(p)$ may be positive or negative. This is because a report by one firm can cause either an increase or decrease in the remaining firms' preemptive motive depending on the curvature of the k_n. To illustrate this, it is helpful to study two contrasting examples. First, consider a three firm setting $(N = 3)$ where $k_1 > k_2 = k_3$ and $\beta \in (k_3, k_1)$. In this case, firms have a preemptive motive as long as nobody has yet reported, but this motive disappears once at least one firm has reported since the firm's order will no longer impact its market share. So here, a change in order reduces the incentive to fake: $b_2(p) - b_1(p) < 0$. Next, consider the same example but now assume that $k_1 = k_2 > k_3$. In this case, all else equal, the first and second firm to report enjoy the same market share. So, firms face no cost to preemption as long as nobody has reported yet. Instead, this cost materializes as soon as the first report has been made. So in this case, a change in order increases the incentive to fake: $b_2(p) - b_1(p) > 0$.

Unlike the change in order, observational learning causes an unambiguous increase in faking. This is stated as Corollary 2.

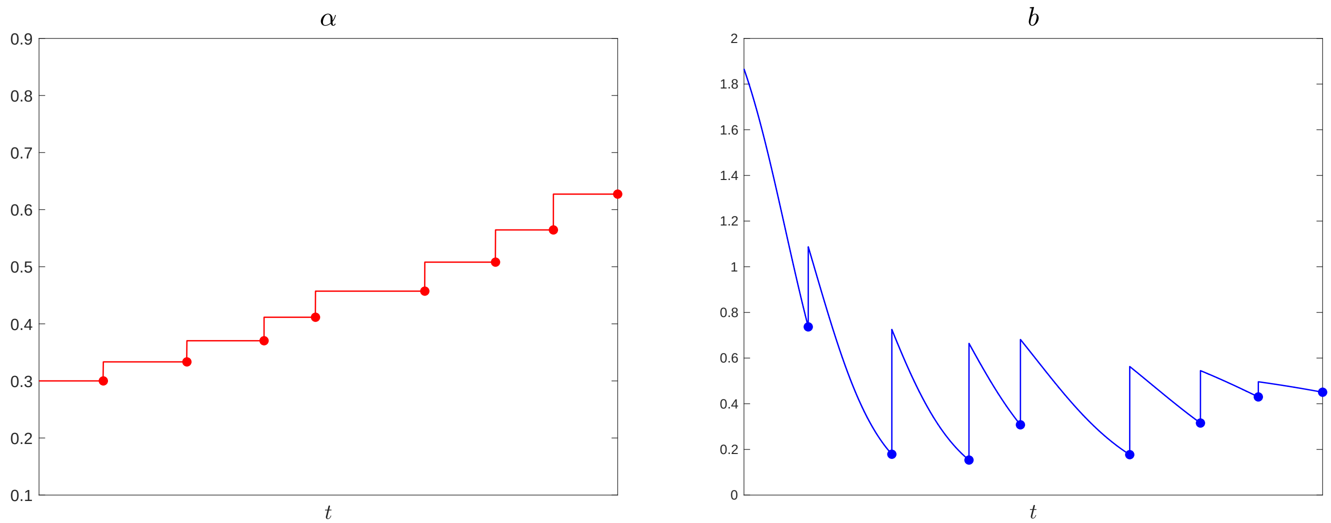

Figure 2: Simulation of crediblity (α) and the hazard rate of faking (b), over the course of a game when $k_N > \beta$. Upwards jumps in b illustrate the copycat effect.

Corollary 2. *For any $n < N$, $b_{n+1}(\tilde{p}) - b_{n+1}(p) \geq 0$, and $b_{n+1}(\tilde{p}) - b_{n+1}(p) > 0$ whenever $b_{n+1}(\tilde{p}) > 0$.*

This is an immediate corollary of Proposition 3 and Proposition 4, which establish that whenever a firm fakes, $b_n(p(t))$ is decreasing in t. Because the common belief $p(t)$ is decreasing in t, this means $b_n(p)$ is increasing in p. That is, a jump in the common belief implies an increase in faking. There is intuition for this as well: all else equal, a firm that is more optimistic has a greater incentive to fake because a higher p corresponds to a lower risk of error and higher risk of preemption, which both make faking more valuable.

While observational learning will always cause faking to increase, the ambiguous effect of order means that the net effect is also ambiguous, i.e., a report does not always result in the copycat effect. However, the copycat effect always occurs when the common belief is sufficiently low. I formalize this as Corollary 3.

Corollary 3. *Suppose $n < N$ and $k_{n+1} > \beta$. There exists a $\overline{p} > 0$ such that for all $p < \overline{p}$, $b_{n+1}(\tilde{p}) - b_n(p) > 0$.*

This result is connected to the fact that the magnitude of observational learning, $\tilde{p} - p$, is decreasing in the starting belief p. This is true for two reasons. First, a high pre-report belief p leaves little room for the belief to increase further. Second, reports made when p is high are less credible, and thus have less impact on the common belief. This negative correlation between the common belief and observational learning means that the positive effect of observational learning on faking is salient when p is low. Indeed, when p is sufficiently small, observational learning is substantial enough to give rise to the copycat effect.

The copycat effect has important implications for the behavior of firms. It implies that in the aftermath of an opponent report, a firm is more likely to report not because they have also uncovered the truth, but because they are faking. If a report had not been made, however, the firms' hazard rate of faking would continue to decrease. That is, the copycat effect is consistent with herding on firms' decision to report. Furthermore, because firm faking increases immediately and then starts its gradual decline, a new report is most likely in the immediate aftermath of an opponent report. That is, this behavior is consistent with herding on the timing of reports, as empirically documented by Cagé et al. (2020). While one might conceive that such herding might be the result of firms receiving correlated signals, the copycat effect demonstrates a strategic motive for such behavior.

5. Media mergers

In this section, I consider the implications of this model for media mergers. I find that while a merger may improve credibility early on in the news cycle, this comes at the expense of lower-credibility reporting later in the news cycle.

Formally, I compare the equilibrium under $N \geq 2$ firms (*pre-merger*) to that under a monopoly (*post-merger*), holding fixed the market's total ability to learn and total maximal market share. I.e., I assume that each firm has ability λ pre-merger and the monopolist has ability $N\lambda$ post-merger. This normalization is motivated by the fact that merging news firms ostensibly combine their news rooms, and thus their capacities for research. Further, I assume the n^{th} firm enjoys a maximal market share k_n pre-merger, while the monopolist enjoys maximal market shares $k^m \equiv \sum_{n=1}^{N} k_n$ post-merger. This is consistent with the notion that merging news firms combine their consumer bases.[12] Finally, I assume that the cost of error (β) and the prior about the story (p_0) is the same pre- and post-merger.

The merger affects reporting by both changing the credibility of the first report and eliminating the possibility of succeeding reports. Let us begin by considering the former, i.e. the impact of the merger on the first news report. It follows from the above characterization that if the prior about the story being true (p_0) is sufficiently high, reporting may be more credible early on in the news cycle. However, later reports will always be less credible post-merger whenever not in a winner-takes-all setting. This second point is formalized as Corollary 4.

[12] For a formal justification, please see the payoff function microfoundation.

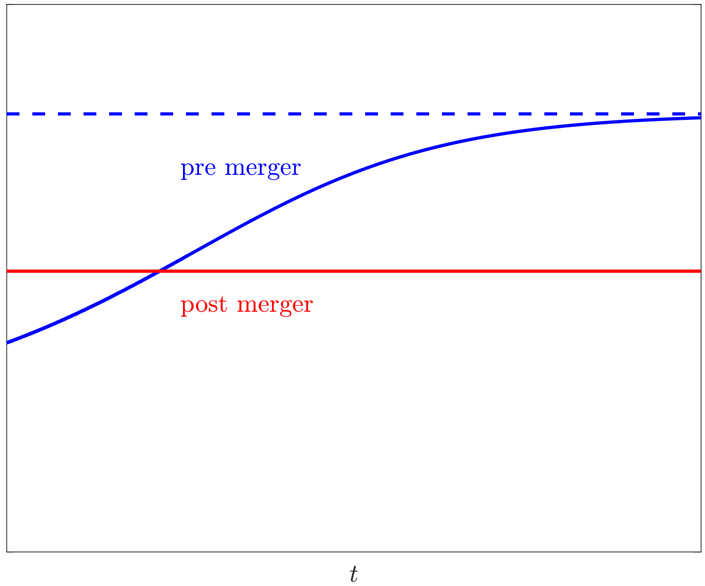

Figure 3: Simulation of crediblity of first report (α_1) pre-merger (with 3 firms) and post-merger as function of time of report.

Corollary 4. *If $\beta \in (k_N, k_1)$, and $k_2 > 0$, there exists a $\overline{p} > 0$ such that $\alpha_1^m(p) < \alpha_1(p)$ for all $p < \overline{p}$, where α and α^m denote the pre- and post-merger equilibrium credibility, respectively.*

These findings are illustrated by Figure 3, which plots a simulation of credibility in the market conditional on the time of the first report, assuming that $\beta \in (k_1, k_N)$. Here, reports are more credible post-merger for small t (i.e., with little time for research), but are less credible once the common belief falls below a threshold. This is due to the fact that the merger affects two changes to the market. First, any preemptive motive firms may have faced before is eliminated. This elimination of the preemptive motive reduces the incentive to fake, and is why post-merger credibility may be higher when the common belief is still relatively high. But in addition to this, the post-merger firm also enjoys a greater maximal market share. All else equal, this makes faking more profitable, and thus deteriorates credibility. While both changes occur simluateneously, the credibility-improving effects of eliminating the preemptive motive disappears as the common belief falls. Furthermore, such credibility-improving effects are limited to instances where firms face a preemptive motive: because there is no cost to preemption when $\beta \leq k_N$, post-merger credibility will be lower no matter when the report is made in this case.

This result has implications for the impact of media mergers on news quality. While previous research has argued that media mergers can exacerbate bias (Anderson and McLaren (2012)) and ideological persuasion (Balan, DeGraba, and Wickelgren (2003)), Corollary 4 suggests that the effect on factual errors is more nuanced. Mergers can improve credibility by eliminating firms' preemptive motive, but such improvement can only occur early on in a news cycle and if firms are sufficiently optimistic about the story to begin with. Otherwise, market consolidation will cause credibility to suffer. Furthermore, beyond affecting the quality of the first report, the merger comes at the cost of reports by succeeding firms. Although such succeeding reports may suffer in credibility due to the copycat effect, they nonetheless serve as additional signals that are lost due to the merger.

6. Comparative statics

In this section, I consider how news credibility changes with the parameters of the model. These findings are stated as Proposition 5.

Proposition 5. *In any equilibrium, for any (p, n), $\alpha_n(p(t))$ is*

(a) weakly increasing in β, and strictly so whenever $\alpha_n(p(t)) < 1$.

(b) weakly increasing in λ, and strictly so for $t > 0$ whenever $\alpha_n(p(t)) < 1$ and $k_N < \beta$.

(c) weakly decreasing in N, and strictly so whenever $\alpha_n(p(t)) < 1$, when $t \in [0, \bar{t}]$ for some $\bar{t} > 0$.

Part (a) states that no matter when a firm reports, it will be more credible under high β. This result is intuitive: a higher ex-post cost of error means firms are less likely to fake, and thus more credible. This is a consequence of the firm's equilbrium incentives: a higher β makes faking more costly. This will either induce the firm to resort to truth telling or, to preserve indifference, require that it is compensated for this costlier faking with higher credibility.

Now, let us consider the comparative static on λ. To understand what is driving this result, let us first note that at any belief p the firm may hold, a change in λ will have no effect on $\alpha_n(p)$ in equilibrium. This is due to the fact that λ does not enter the boundary value problem which dictates the firm's credibility, and thus changes in λ do not affect $\alpha_n(p)$. However, changes in λ will affect the common belief $p(t)$: under a higher λ, firms learn about the state more quickly, and thus firms will be more pessimistic about the story's validity at any time $t > 0$. This greater pessimism about the story translates to a higher expected cost of erring, which makes faking more costly. As was true of the comparative static on β, this increased cost of faking must be counterbalanced by higher credibility $\alpha_1(p(t))$ at every time $t > 0$ to ensure indifference holds.

Let us finally consider the comparative static on the total number of firms, N. This exercise is distinct from the merger analysis in the previous section. With this comparative

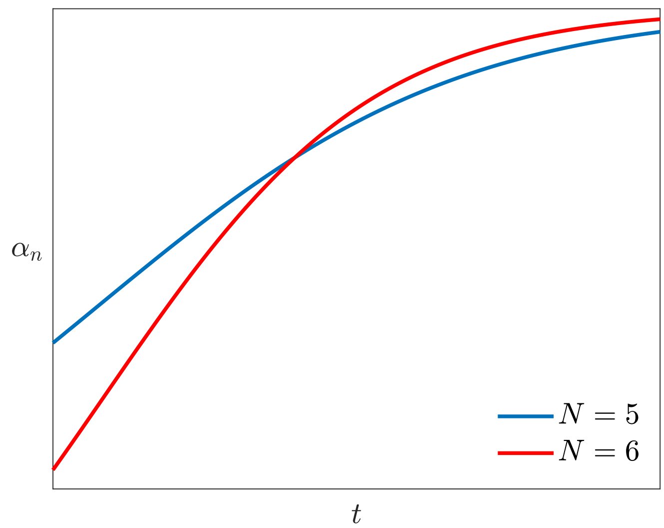

Figure 4: A simulation of $\alpha_n(p(t))$ when $N = 5$ (blue line) and $N = 6$ (red line). For the remaining parameter values, the following specifications were made: $\beta = 0.5$, $p_0 = 0.7$, $\lambda = 1$, $k_n = 0.7^{(N-n)}$.

static, we are studying the marginal impact of an additional firm entering the market. In particular, I do not hold fixed the market's aggregate learning ability. Rather, I assume that this additional firm adds to the total learning ability of the market. In doing so, one can study the effects of firm entry. Proposition 5 states that firm entry deteriorates the credibility of the first report, but only early on in the news cycle. In fact, it may result in an improvement in credibility later on in the news cycle. This phenomenon is illustrated by Figure 4. While the addition of a firm lowers credibility for low t, it improves credibility after enough time has passed.

This result is due to the fact that an additional firm affects two separate changes to the market. First, each firm faces greater competition, and thus a greater risk of being preempted. Second, an additional firm also increases the market's ability learn observationally. This change is captured by the comparative static on λ. Thus, the effect of an additional firm can be understood as the combination of two countervailing forces: higher competition which deteriorates credibility, and a greater ability to learn, which improves credibility.

To understand why the credibility-diminishing effect of higher competition dominates when t is small, we must compare the relative magnitudes of these the two forces. An increase in learning ability has a negligible impact on credibility for early reports. This is due to the fact that firms learn gradually over time, and thus it takes time for differences in learning ability to substantially impact firms' beliefs. Meanwhile, an increase in competition will have a non-negligible impact on credibility even when $t = 0$. For this reason, the impact of higher competition dominates when t is small, resulting in a net reduction in credibility. However, as time passes and the effect of faster learning grows, a reversal may take place, i.e., there may be a net improvement in credibility. Such a scenario is precisely what is depicted by Figure 4.

7. Extension: heterogeneous ability

I now consider an extension in which firms have heterogeneous learning abilities. This will shed light on how a firm's credibility correlates with its ability in equilibrium.

The extended model is identical to the model above except for three changes. First, rather than assuming that each firm is endowed with the same ability λ, I assume that each firm i is endowed with an firm-specific ability λ^i, which is common knowledge. Second, for tractibility, I restrict attention to a winner-takes-all setting: i.e., I assume $k_n = 0$ for all $n > 1$. Finally, I relax the equilibrium symmetry assumption. Accordingly, I let α^i denote the credibility of firm i.

I obtain an intuitive result: firms with higher ability are more credible in equilibrium.

Proposition 6. *For all (i, j) such that $\lambda^i < \lambda^j$, $\alpha_1^i(p(t)) \leq \alpha_1^j(p(t))$. Furthermore, this inequality is strict whenever $\alpha_1^i(p(t)) < 1$.*

Proposition 6 states that regardless of when a report is made, a firm with higher ability is weakly more credible, and strictly so whenever firms are not fully truthful. Let us consider why this correlation arises. First, note that high ability firms are able to confirm a story more quickly and thus, all else equal, pose a greater preemptive threat in equilibrium. This in turn implies that in comparison to a high-ability firm, a low-ability firm faces a greater preemptive threat. Thus, the low-ability firm finds immediate faking more advantageous. In light of this, the firms' credibilities must adjust in such a way to preserve their respective indifference conditions. This is achieved endogenously by means of a lower credibility for the low-ability firm, which ensures that it has less to gain from faking.

8. Conclusion

This paper presents a dynamic model of breaking news, accounting for both the preemptive motive firms face and consumers' preference for credibility. I find that errors are driven by three features of the breaking news environment. The first is an inability for firms to commit to truthful reporting, which can give rise to errors even under a monopoly.

Errors are exacerbated in an N-firm setting due to competition and observational learning: competition incentivizes hasty reporting by giving rise to a preemptive motive while observational learning causes existing errors to propagate. The equilibrium also exhibits rich dynamics in firm behavior. First, firms become gradually more truthful over time as long as no new reports are made. Furthermore, a firm's credibility gradually increases whenever preemptive motives are at play which mitigates this preemptive motive. This improvement in credibility incentivizes firms to delay reporting, and thus counteracts the haste-inducing effects of preemption. Dynamics also take the form of discrete and persistent changes in the firm's behavior and credibility which are triggered by a rival report. In particular, I document a copycat effect, where a report by one firm induces a surge in faking by other firms in the market, behavior that is consistent with clustering in both errors and valid reports. More broadly, this model provides insight into how preemptive concerns can affect the quality of information provided by experts. To understand how preemption impacts information provision generally is a topic that warrants further investigation.

References

Simon P Anderson and John McLaren. Media mergers and media bias with rational consumers. *Journal of the European Economic Association*, 10(4):831–859, 2012.

Giovanni Andreottola, Antoni-Italo de Moragas, et al. *Scandals, Media Competition and Political Accountability*. CSEF, Centre for Studies in Economics and Finance, Department of Economics . . . , 2020.

David J Balan, Patrick DeGraba, and Abraham L Wickelgren. Media mergers and the ideological content of programming. *Bureau of Economics FTC*, 2003.

Michael R Baye, Dan Kovenock, and Casper G De Vries. The All-Pay Auction with Complete Information. *Economic Theory*, 8(2):291–305, 1996.

Catherine Bobtcheff, Raphaël Levy, and Thomas Mariotti. Negative results in science: Blessing or (winner's) curse? 2022.

Raphael Boleslavsky and Curtis R Taylor. Make It 'Til You Fake It. *Economic Research Initiatives at Duke (ERID) Working Paper Forthcoming*, 2020.

Julia Cagé, Nicolas Hervé, and Marie-Luce Viaud. The production of information in an online world. *The Review of economic studies*, 87(5):2126–2164, 2020.

Pew Research Center. For Local News, Americans Embrace Digital but Still Want Strong Community Connection. *Pew Research Center*, 2019.

Christophe Chamley and Douglas Gale. Information Revelation and Strategic Delay in a Model of Investment. *Econometrica: Journal of the Econometric Society*, pages 1065–1085, 1994.

Yeon-Koo Che and Johannes Hörner. Recommender Systems as Mechanisms for Social Learning. *The Quarterly Journal of Economics*, 133(2):871–925, 2018.

Chia-Hui Chen, Junichiro Ishida, and Arijit Mukherjee. Pioneer, early follower or late entrant: Entry dynamics with learning and market competition. *European Economic Review*, 152:104360, 2023.

Heng Chen and Wing Suen. Competition for Attention and News Quality. Technical report, Working paper, 2019.

Drew Fudenberg and Jean Tirole. Preemption and Rent Equalization In the Adoption of New Technology. *The Review of Economic Studies*, 52(3):383–401, 1985.

Drew Fudenberg, Richard Gilbert, Joseph Stiglitz, and Jean Tirole. Preemption, Leapfrogging and Competition In Patent Races. *European Economic Review*, 22(1):3–31, 1983.

Simone Galperti and Isabel Trevino. Coordination Motives and Competition for Attention in Information Markets. *Journal of Economic Theory*, 188:105039, 2020.

Matthew Gentzkow and Jesse M Shapiro. Competition and Truth in the Market for News. *Journal of Economic Perspectives*, 22(2):133–154, 2008.

Jeffrey Gottfried, Mason Walker, and Amy Mitchell. Americans See Skepticism of News Media as Healthy, Say Public Trust in the Institution Can Improve. *Pew Research Center*, 2020.

Steven R Grenadier. Information revelation through option exercise. *The Review of Financial Studies*, 12(1):95–129, 1999.

Ken Hendricks, Andrew Weiss, and Charles Wilson. The War of Attrition in Continuous Time with Complete Information. *International Economic Review*, pages 663–680, 1988.

Hugo A Hopenhayn and Francesco Squintani. Preemption Games with Private Information. *The Review of Economic Studies*, 78(2):667–692, 2011.

Rida Laraki, Eilon Solan, and Nicolas Vieille. Continuous-Time Games of Timing. *Journal of Economic Theory*, 120(2):206–238, 2005.

Charles Lin. Speed Versus Accuracy in News. 2014.

Giuseppe Moscarini and Francesco Squintani. Competitive experimentation with private information: The survivor's curse. *Journal of Economic Theory*, 145(2):639–660, 2010.

Pauli Murto and Juuso Välimäki. Learning and information aggregation in an exit game. *The Review of Economic Studies*, 78(4):1426–1461, 2011.

Ayush Pant and Federico Trombetta. The Newsroom Dilemma. *Available at SSRN 3447908*, 2019.

V Wang. ABC Suspends Reporter Brian Ross Over Erroneous Report About Trump. *The New York Times*, 2017.

Appendix A Microfoundation for market share

In the main text, I assumed that the the firm's market share from reporting a story is $k_n\alpha$. Here I provide a microfoundation for this.

Let $\mathcal{N} \equiv \{1, ..., N\}$ denote the set of news firms. Suppose there is a mass $K > 0$ of consumers, who are indexed by x. Each consumer x *subscribes* to some subset S_x of the firms. I.e., for all x, $S_x \subseteq \mathcal{N}$. Let S_x denote consumer x's *subscription set*. Fixing any $S \subseteq \{1, ..., N\}$, let $m(S)$ denote the mass of consumers x such that $S_x = S$, where $\sum_{S \in 2^{\mathcal{N}}} m(S) = K$. Assume that the mass of consumers with a given subscription set does not depend on the identity of the firms within that set, but only on the number of firms in the set. Formally, suppose that there exists $\gamma_0, \gamma_1, ..., \gamma_N \geq 0$ such that

$$m(S) = \gamma_n \text{ if and only if } |S| = n, \text{ where } \sum_{n=1}^{N} \gamma_n \binom{N}{n} = K.$$

Define i's *market share* to be the mass of consumers who read the story. We assume that a consumer reads a story if she both *considers* the story, and finds it optimal to read it. To formalize this, let $\hat{S} \subseteq \mathcal{N}$ denote the set of firms who reported before i. A consumer x will *consider* a story if and only if:

1. The firm is in the consumer's subscription set, i.e., $i \in S_x$.
2. The consumer has not previously considered the story. I.e., $j \notin S_x$ for all $j \in \hat{S}$.

The mass of consumers who consider firm i's story is then given by

$$\sum_{j=1}^{N-n} \binom{N-n}{j} \gamma_{j+1} \equiv k_n,$$

where n is the order of i's report. Next, suppose consumer x faces a cost c_x of reading a story. Suppose that c_x is i.i.d. across x, that for any x, c_x is uniformly distributed on $[0,1]$, and that c_x is independent of x's consideration set. Then x's payoff from reading a story is $\mathbb{I}[\theta = 1] - c_x$. That is, the consumer will incur a cost c_x from reading the story, and a benefit of 1 only if the story is true. Meanwhile, the consumer's payoff from not reading a story is $\mathbb{I}[\theta = 0]$. Namely, the consumer enjoys a payoff of 1 from refusing to story that is untrue. Assuming consumers maximize expected utility, x will read the story if and only if

$$\alpha + (1-\alpha)p - c_x \geq (1-\alpha)(1-p) \Leftrightarrow c_i \leq \alpha$$

where α is the credibility of i's story. Thus i's market share is $k_n \alpha$.

Appendix B Equilibrium credibility

Here, I justify equation (4) by showing that it is the limit of Bayes-consistent beliefs under a discrete approximation of the game presented in Section 2. To this end, for any $\varepsilon > 0$, let the *ε-approximation* of the game be identical to the game presented in section (2), except with the following modification: any report made by a firm on $[0, \varepsilon]$ is observed by all other players (including the consumer) at ε. That is, rather than observing t_i, the players observe $\tilde{t}_i$, where

$$\tilde{t}_i \equiv max\{t_i, \varepsilon\}$$

At any (p, n) that is on-path, let $\alpha_n^\varepsilon(p)$ denote the firm's credibility, i.e., the consumer's belief that $s_i \leq \varepsilon$ given that $\tilde{t}_i = \varepsilon$, under the ε-approximation of the game. Let α_n denote the right-limit of the α_n^ε. Then:

$$\alpha_n(p) \equiv \lim_{\varepsilon \to 0+} \alpha_n^\varepsilon(p)$$

I now establish that $\alpha_n(p)$ is given by (4) at any (p, n) on-path.

Claim 3. *For any (p, n) on-path,*

$$\alpha_n(p) = \begin{cases} \frac{\lambda p}{\lambda p + b_n(p)} & \text{if } F_{p,n}(0) = 0 \\ 0 & \text{if } F_{p,n}(0) > 0 \end{cases}$$

Proof. For any $\varepsilon > 0$, it follows from Bayes Rule that

$$\alpha_n^\varepsilon(p) = \frac{p(1-e^{-\lambda\varepsilon})}{p(1-e^{-\lambda\varepsilon}) + F_{p,n}(\varepsilon)e^{-\lambda\varepsilon}}.$$

If $F_{p,n}(0) = 0$, it follows from L'Hôpital's Rule that:

$$\lim_{\varepsilon \to 0+} \alpha_n^\varepsilon(p) = \frac{\lambda p}{\lambda p + b_n(p)}$$

If $F_{p,n}(0) > 0$, it follows from the right-continuity of $F_{p,n}$ that

$$\lim_{\varepsilon \to 0+} \alpha_n^\varepsilon(p) = \frac{0}{0 + \lim_{\varepsilon \to 0+} F_{p,n}(\varepsilon)} = 0.$$

□

Appendix C Equilibrium characterization proofs

Proof of Lemma 1. Let us begin by showing that at all (p, n) on-path such that $p < 1$, $F_{p,n}$ is continuous at 0. To this end, suppose by contradiction that $F_{p,n}$ is discontinuous at 0. By the right-continuity of $F_{p,n}$, this implies that $F_{p,n}(0) > 0$. Because (p, n) is on path, by (4), $\alpha_n(p) = 0$. Furthermore, it follows by definition that $p^i(0) = p$. Recalling that we are restricting attention to symmetric equilibria, let Ψ denote the first-report distribution at (p, n) under the equilibrium strategy profile $F_{p,n}$. Because $F_{p,n}(0) > 0$, $\Psi^i(0) > 0$ for all i who have not yet reported.

Now define the following deviation $\hat{F}_{p,n}$. This strategy is identical to $F_{p,n}$, except that all the mass that $F_{p,n}$ places on 0 is shifted to ∞:

$$\hat{F}_{p,n}(s) = \begin{cases} F_{p,n}(s) - F_{p,n}(0) & \text{if } s < \infty \\ 1 & \text{if } s = \infty \end{cases}$$

Now, fix some i who has not yet reported. Let $\hat{\Psi}$ denote the first-report distribution at (p, n) under the strategy profile where i plays $\hat{F}_{p,n}$ and all $j \neq i$ play $F_{p,n}$. By definition, for all $s \geq 0$,

$$\hat{\Psi}^i(s) = \Psi^i(s) - \Psi^i(0).$$

Then,

$$\int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\hat{\Psi}^i(s) = \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\Psi^i(s) + \beta(1-p^i(0))\Psi^i(0)$$
$$> \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\Psi^i(s).$$

Again by definition, for all $s \geq 0$,

$$\hat{\Psi}^{-i}(s) = \Psi^{-i}(s) + X(s),$$

where

$$X(s) \equiv \Psi^i(0)[p\int_0^s (1-F_{p,n})^{N-n-1}(1-\hat{F}_{p,n}(r))e^{-\lambda r(N-n)}d(e^{-\lambda r}(F_{p,n}(r)-1))$$
$$+(1-p)\int_0^s (1-F_{p,n}(r))^{N-n-1}(1-\hat{F}_{p,n}(r))dF_{p,n}(r)].$$

Then, we have

$$\int_0^\infty V_{p^{-i}(s),n+1}d\hat{\Psi}^{-i}(s) - \int_0^\infty V_{p^{-i}(s),n+1}d\Psi^{-i}(s) = \int_0^\infty V_{p^{-i}(s),n+1}dX(s) \geq 0.$$

where the final inequality follows from the fact that $X(s)$ is increasing in s and $V_{p^{-i}(s),n+1} \geq V_{p^{-i}(s),n+1}(\delta_\infty) \geq 0$.

Combining the above two inequalities we have

$$V_{p,n}(\hat{F}_{p,n}) = \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\hat{\Psi}^i(s) + (N-n)\int_0^\infty V_{p^{-i}(s),n+1}d\hat{\Psi}^{-i}(s)$$
$$> \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\Psi^i(s) + (N-n)\int_0^\infty V_{p^{-i}(s),n+1}d\Psi^{-i}(s) = V_{p,n}(F_{p,n}).$$

Thus, i can profitably deviate at (p,n). Contradiction.

It remains to show that continuity applies at all t were (p,n) on-path such that $p < 1$. Suppose by contradiction that it is not. Let t denote the time at which there is a discontinuity. Because $F_{p,n}$ is increasing and right-differentiable

$$\lim_{r\to t-} F_{p,n}(r) < F_{p,n}(t).$$

By (3),

$$F_{p(t),n}(0) = \frac{F_{p,n}(t) - \lim_{r\uparrow t} F_{p,n}(r)}{1 - \lim_{r\uparrow t} F_{p,n}(r)} > 0.$$

Thus, $F_{p(t),n}$ is discontinuous at 0, contradicting the above. $\square$

Lemma 3. *For any (p, n) on-path,*

- $\alpha_n(p) \geq \underline{\alpha}_n(p) \equiv \min\{\beta(1-p)/k_n, 1\}$
- $F'_{p,n}(0+) \leq \overline{f} \equiv \lambda p(\frac{1}{\underline{\alpha}_n(p)} - 1)$.

Proof of Lemma 3. We begin by showing the first point above. The second point follows by definition of $\alpha_n(p)$.

First, suppose by contradiction that there exists a (p, n) on-path such that

$$\alpha_n(p) < \min\{\beta(1-p)/k_n, 1\}.$$

Recalling that $p(s)$ is given by (2), I begin by showing that for all s sufficiently small, $(p(s), n)$ is on-path. Suppose not by contradiction. Since (p, n) is on-path by assumption, this implies that $F_{p,n}(s) = 1$, which contradicts Lemma 1. It thus follows from (4), combined with the piecewise twice differentiability and right-differentiability of $F_{p,n}$, that $\alpha_n(p(s))$ is continuous in some right-neighborhood of $s = 0$. Thus, there exists an $\varepsilon > 0$ such that for all $s \in [0, \varepsilon]$,

$$k_n \alpha_n(p(s)) < \beta(1-p).$$

Next, I claim that $F_{p,n}(\varepsilon) > 0$. Suppose this is not true by contradiction. Then, it follows that $F_{p,n}(s) = 0$ for all $s \in [0, \varepsilon]$, implying by definition of α that $\alpha_n(p) = 1$, contradicting the assumption that $\alpha_n(p) < 1$.

Now, define the following deviation $\tilde{F}_{p,n}$, which shifts the mass $F_{p,n}$ places on $[0, \varepsilon]$ to ∞:

$$\tilde{F}_{p,n}(s) = \begin{cases} 0 & \text{if } s \in [0, \varepsilon] \\ F_{p,n}(s) - F_{p,n}(\varepsilon) & \text{if } s \in (\varepsilon, \infty) \\ 1 & \text{if } s = \infty. \end{cases}$$

The admissibility (i.e., right-continuity and piecewise twice-differentiability) of $\tilde{F}_{p,n}$ follows from the admissibility of $F_{p,n}$. We now wish to show that $\tilde{F}_{p,n}$ is a profitable deviation at (p, n). Let Ψ denote the first-report distribution under the strategy profile where all players play $F_{p,n}$, and let $\tilde{\Psi}$ denote the first-report distribution under the strategy profile where i plays $\tilde{F}_{p,n}$ and all $j \neq i$ play $F_{p,n}$.

By definition of Ψ,

$$\tilde{\Psi}^i(s) = \Psi^i(s) - X(s),$$

where

$$X(s) = \begin{cases} p\int_0^s e^{-\lambda r(N-n)}(1-F_{p,n}(r))^{N-n}d(e^{-\lambda r}(F_{p,n}(r)-1)) + (1-p)\int_0^s(1-F_{p,n}(r))^{N-n}dF_{p,n}(r) & \text{if } s \in [0,\varepsilon] \\ X(\varepsilon) & \text{if } s > \varepsilon. \end{cases}$$

Now, note that $X(s)$ is weakly increasing in s. Note further that because $F_{p,n}(\varepsilon) > 0$, it follows that $F_{p,n}(s)$ strictly increases on $[0,\varepsilon]$. Thus, $X(s)$ is strictly increasing at some $s \in [0,\varepsilon]$. Now, by the above definition:

$$\int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\tilde{\Psi}^i(s) - \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\Psi^i(s)$$
$$= -\int_0^\varepsilon [k_n\alpha_n(p(s)) - \beta(1-p(s))]dX(s) > 0.$$

where the strict inequality follows from the fact that $X(s)$ is strictly increasing on $[0,\varepsilon]$ and the above-established fact that $k_n\alpha_n(p(s)) < \beta(1-p(s))$ for all $s \in [0,\varepsilon]$.

Next, let us consider $\tilde{\Psi}^{-i}(s)$. It again follows from the definition of Ψ that

$$\tilde{\Psi}^{-i}(s) = \Psi^{-i}(s) - Y(s),$$

where

$$Y(s) = -p\int_0^s [e^{-\lambda r}(1-F_{p,n}(r))]^{n-2}F(\min\{r,\varepsilon\})d(e^{-\lambda r}(F_{p,n}(r)-1)) -$$
$$(1-p)\int_0^s (1-F_{p,n}(r))^{n-2}F_{p,n}(\min\{r,\varepsilon\})dF_{p,n}(r).$$

Thus,

$$\int_0^\infty V_{p^{-i}(s),n+1}d\tilde{\Psi}^{-i}(s) - \int_0^\infty V_{p^{-i}(s),n+1}d\Psi^{-i}(s) = \int_0^\infty V_{p^{-i}(s),n+1}dY(s) \geq 0.$$

where the final inequality follows from the fact that $Y(s)$ is weakly increasing in s and $V_{p^{-i}(s),n+1} \geq 0$. Combining the previous two inequalities, we obtain that

$$V_{p,n}(\tilde{F}_{p,n}) > V_{p,n}(F_{p,n}),$$

and thus i can profitably deviate at (p,n). Contradiction. □

Proof of Lemma 2. Assume that $\alpha_n(p) < 1$. By the right twice-differentiability of $F_{p,n}$, and by (4), that $\alpha_n(p(s))$ is right-continuous in s. Thus, there exists an $\varepsilon > 0$ and $d > 0$ such that

$$\alpha_n(p(s)) < 1 - d \text{ for all } s \in [0, \varepsilon).$$

I claim that for all $s \in [0, \varepsilon)$, $V_{p,n} = V_{p,n}(\delta_s)$. Suppose by contradiction that for some $\tilde{s} \in [0, \varepsilon)$,

$$V_{p,n}(\delta_s) < V_{p,n}.$$

Now, I show that $V_{p,n}(\delta_s)$ is right-continuous in s. By definition,

$$V_{p,n}(\delta_s) = \int_0^s k_n \alpha_n(p(r)) d\Psi^i(r) + (N - n) \int_0^s V_{p^i(r),n} d\Psi^{-i}(r) + \\ (1 - \sum_j \Psi^j(s))[k_n \alpha_n(p(s)) - \beta(1 - p(s))],$$

where $\Psi^j(s)$ is the first-report distribution that arises when i plays δ_∞ and all $j \neq i$ play $F_{p,n}$. The right-continuity of $V_{p,n}(\delta_s)$ with respect to s then follows from the absolute continuity of Ψ^j (which follows from Lemma 1), and the right-continuity of $\alpha_n(p(s))$ with respect to s, which follows from the right-continuity of $F_{p,n}(s)$ by assumption.

Given the right continuity of $V_{p,n}(\delta_s)$, there exists some $\varepsilon' \in (0, \varepsilon - \tilde{s})$ and $x > 0$ such that

$$V_{p,n} - V_{p,n}(\delta_r) > x \text{ for all } r \in [\tilde{s}, \tilde{s} + \varepsilon'].$$

Now I claim that there must exist some $s^* \in [0, \infty]$ such that $V_{p,n} = V_{p,n}(\delta_{s^*})$. Suppose by contradiction that $V_{p,n} > V_{p,n}(\delta_s)$ for all $s \in [0, \infty]$. It follows from (5) that

$$V_{p,n}(F) = \int_0^\infty V_{p,n}(\delta_s) dF_{p,n}(s) + (1 - \lim_{s \to \infty} F_{p,n}) V_{p,n}(\delta_\infty) < V_{p,n},$$

where the strict inequality follows from the assumption that $V_{p,n} > V_{p,n}(\delta_s)$ for all s. Thus, F cannot be an equilibrium strategy. Contradiction.

Now, define the following deviation $\tilde{F}$. This strategy is identical to F, except $\tilde{F}_{p,n}$ shifts all the mass from $[s, s + \varepsilon']$ to s^*. Specifically, when $s^* < \tilde{s}$:

$$\tilde{F}_{p,n}(t) = \begin{cases} F_{p,n}(t) + F_{p,n}(\tilde{s} + \varepsilon) - F_{p,n}(\tilde{s}) & \text{if } t \in [s^*, \tilde{s}] \\ F_{p,n}(\tilde{s} + \varepsilon) & \text{if } t \in (\tilde{s}, \tilde{s} + \varepsilon'] \\ F_{p,n}(t) & \text{otherwise.} \end{cases}$$

Meanwhile, when $s^* > \tilde{s} + \varepsilon'$:

$$\tilde{F}_{p,n}(t) = \begin{cases} F_{p,n}(s) & \text{if } t \in [\tilde{s}, \tilde{s} + \varepsilon] \\ F_{p,n}(t) - [F_{p,n}(\tilde{s} + \varepsilon') - F_{p,n}(\tilde{s})] & \text{if } t \in (\tilde{s} + \varepsilon', s^*) \\ F_{p,n}(t) & \text{otherwise.} \end{cases}$$

Now, by definition:

$$V_{p,n}(\tilde{F}) = V_{p,n}(F) + \int_{\tilde{s}}^{\tilde{s}+\varepsilon'} [V_{p,n}(\delta_{s^*}) - V_{p,n}](\delta_r) dF_{p,n}(r) \geq V_{p,n}(F) + x\varepsilon' > V_{p,n}(F_{p,n}).$$

Thus, $\tilde{F}$ is a profitable deviation. Contradiction.

It remains to show that $V_{p,n} = V_{p,n}(\delta_\infty)$. Suppose by contradiction that $V_{p,n} > V_{p,n}(\delta_\infty)$. It follows that $\lim_{t\to\infty} F_{p,n}(t) = 0$, because otherwise, the firm could profitably deviate by placing no mass on $t = \infty$. This implies that for some $s \in (0, \infty]$,

$$\lim_{t\to s-} b_n(p(t)) = \infty \Rightarrow \lim_{t\to s-} \alpha_n(p(t)) = 0,$$

which contradicts Lemma 3. $\square$

Lemma 4. *$\alpha_n(p(s))$ is continuous in s for all (p, n) on path such that $s > 0$.*

Proof of Lemma 4. Fix a (p, n) on-path. I first show that for all $s \geq 0$,

$$\alpha_n(p(s)) = \frac{\lambda p(s)}{\lambda p(s) + \frac{F'_{p,n}(s+)}{1-F_{p,n}(s)}} \tag{9}$$

It follows from Lemma 3 that $(p(s), n)$ is on-path for all $s \geq 0$. Thus, by Lemma 1, $F_{p(s),n}(0) = 0$, and by (4)

$$\alpha_n(p(s)) = \frac{\lambda p(s)}{\lambda p(s) + F'_{p(s),n}(0+)}.$$

Next, it follows from (3) that

$$F'_{p(s),n}(0+) = \frac{F'_{p,n}(s+)}{1 - F_{p,n}(s)}.$$

Combining the previous two equations yields (9). It thus follows from the right-differentiability and piecewise twice-differentiability of $F_{p,n}$ that $\alpha_n(p(s))$ is right-continuous in s. It remains to show that $\alpha_n(p(s))$ is left-continuous in s. Suppose by contradiction there exists an s such that $\alpha_n(p(s))$ is left-discontinuous. Then there exists some $d > 0$ such that for all $\varepsilon > 0$, there exists an $s_\varepsilon \in (s-\varepsilon, s)$ such that

$$|\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s))| > d.$$

First consider the case where for all $\varepsilon > 0$, there exists an $s_\varepsilon \in (s-\varepsilon, s)$ such that $\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s)) > d$. I begin by claiming that for all $\varepsilon > 0$,

$$V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}). \tag{10}$$

Note that there exists some $s^* \in (s, \infty]$ such that $V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_{s^*-s_\varepsilon})$. To see why this must hold, suppose not, by contradiction. Then it must be that $F_{p(s_\varepsilon),n}$ places full mass on $[0, s-s_\varepsilon]$, and thus, either Lemma 1 or (3) would be violated. Thus, we have

$$V_{p(s_\varepsilon),n} = \int_0^{s-s_\varepsilon} k_n\alpha_n(p(s_\varepsilon+r))d\Psi^i(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(s_\varepsilon+r),n+1}d\Psi^{-i}(r) +$$
$$(1-\sum_j \Psi^j(s-s_\varepsilon))V_{p(s),n}(\delta_{s^*-s}) = \int_0^{s-s_\varepsilon} k_n\alpha_n(p(s_\varepsilon+r))d\Psi^i(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(s_\varepsilon+r),n+1}d\Psi^{-i}(r)$$
$$+(1-\sum_j \Psi^j(s-s_\varepsilon))V_{p(s),n}(\delta_0) = V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}),$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p(s_\varepsilon),n}$. Note that the equality follows from the fact that $\alpha_n(p(s)) < 1$, and thus by Lemma 2, $V_{p(s),n} = V_{p(s),n}(\delta_0)$. However, note that for all $\varepsilon > 0$,

$$V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = \int_0^{s-s_\varepsilon} k_n\alpha_n(p(s_\varepsilon+r))d\Psi^i(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(s_\varepsilon+r),n+1}d\Psi^{-i}(r)$$
$$+(1-\sum_j \Psi^j(s-s_\varepsilon))[k_n\alpha_n(p(s),n) - \beta(1-p(s))].$$

Because the Ψ^j are absolutely continuous,

$$\lim_{\varepsilon\to 0} V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = k_n\alpha_n(p(s),n) - \beta(1-p(s)).$$

Then, by the assumption that $\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s)) > d$, for all $\varepsilon > 0$ sufficiently small $V_{p(s_\varepsilon),n}(\delta_0) = k_n\alpha_n(p(s_\varepsilon),n) - \beta(1-p(s_\varepsilon)) > V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon})$, contradicting (10).

Next, consider the case where for all $\varepsilon > 0$, $\alpha_n(p(s)) - \alpha_n(p(s_\varepsilon)) > d$. As shown above,

$\lim_{\varepsilon\to 0} V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = V_{p(s),n}(\delta_0)$. Thus, for ε sufficiently small,

$$V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) > k_n\alpha_n(p(s_\varepsilon)) - \beta(1-p(s_\varepsilon)) = V_{p(s_\varepsilon),n}(\delta_0).$$

However, since $\alpha_n(p(s_\varepsilon)) < 1$ for all $\varepsilon > 0$, by Lemma 2, $V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_0)$. Contradiction. $\square$

Proof of Proposition 1. I begin by showing that $\alpha_n(p) = 1$ whenever $k_n < \beta$ and $p \leq p_n^* \equiv \frac{k_n-\beta}{k_n/n-\beta}$. To this end, fix an n, and suppose that $k_n < \beta$. I first show that for all $q < \frac{\beta-k_n}{\beta}$, $\alpha_n(q) = 1$. Note that for all such q

$$V_{q,n}(\delta_0) = k_n\alpha_n(q) - \beta(1-q) \leq k_n - \beta(1-q) < k_n - \beta(1-\frac{\beta-k_n}{\beta}) = 0.$$

Since $V_{q,n} \geq V_{q,n}(\delta_\infty) \geq 0$, it follows $V_{q,n} > V_{q,n}(\delta_0)$. Thus, by Lemma 2, $\alpha_n(q) = 1$. Now, let

$$q_n^* \equiv \sup\{p|\alpha_n(q) = 1 \text{ for all } q < p\}.$$

It follows from the above that $q_n^* \geq \frac{\beta-k_n}{\beta}$. I claim that $q_n^* \leq p_n^*$. Suppose by contradiction that $q_n^* < p_n^*$. By Lemma 4, there exists an $\varepsilon > 0$ such that for all $p \in (q_n^*, q_n^*+\varepsilon)$, $\alpha_n(p) < 1$, and thus, by Lemma 2

$$V_{p,n} = V_{p,n}(\delta_0) = k_n\alpha_n(p) - \beta(1-p).$$

Thus, it follows from Lemma 4 that

$$\lim_{p\to q_n^*+} V_{p,n} = k_n - \beta(1-q_n^*). \tag{11}$$

By definition of V, because by Lemma 1 $F_{p,n}$ is absolutely continuous, it follows that $V_{p,n}(\delta_\infty)$ is as well, and thus:

$$\lim_{p\to q_n^*+} V_{p,n}(\delta_\infty) = V_{q_n^*,n}(\delta_\infty) = \frac{k_n q_n^*}{n}. \tag{12}$$

In order for δ_∞ to not serve as a profitable deviation for $p \in (q_n^*, q_n^*+\varepsilon)$, it must be that for all such p, $V_{p,n}(\delta_0) \geq V_{p,n}(\delta_\infty)$. Taking a limit we obtain that

$$\lim_{p\to q_n^*+} V_{p,n}(\delta_0) \geq \lim_{p\to q_n^*+} V_{p,n}(\delta_\infty).$$

Substituting (11) and (12) above, we obtain that $\frac{k_n q_n^*}{n} \leq k_n - \beta(1-q_n^*)$. However, $k_n \leq \beta$ and $q_n^* < p$ implies that $\frac{k_n q^*}{n} > k_n - \beta(1-q^*)$. Contradiction.

Next, we show that $\alpha_n(p) < 1$ whenever $\beta \leq k_n$ or $p > p_n^*$. To this end, assume $\beta \leq k_n$ or $p > p_n^*$. Assume by contradiction that $\alpha_n(p) = 1$. Also assume by induction that if $n < N$, then the statement holds for $n + 1$.

First, consider the case where $\alpha_n(q) = 1$ for all $q < p$. By (4), this implies that $F_(q, n)'(0) = 0$ for all $q < p$. Furthermore, by Lemma 1, this implies that $F_{p,n}(s) = 0$ for all $s > 0$, i.e., $F_{p,n} = \delta_\infty$. However,

$$V_{p,n}(\delta_0) = k_n - \beta(1-p) > \frac{k_n p}{n} = V_{p,n}(\delta_\infty),$$

where the strict inequality follows from the assumption that either $\beta \leq k_n$ or $p > p_n^*$. Contradiction.

Next, consider the case where $\alpha_n(q) < 1$ for some $q < p$. By Lemma 4, for all $\varepsilon > 0$ sufficiently small, there exists some $\overline{p} < p$ and $\overline{s} > 0$ such that $\alpha_n(\overline{p}) \in (1-\varepsilon, 1)$ and $\alpha_n(q)$ is strictly increasing on $[\overline{p}(\overline{s}), \overline{p}]$. By Lemma 2, there exists some $\Delta \in (0, s)$ such that

$$V_{\overline{p},n}(\delta_\Delta) = V_{\overline{p},n}(\delta_0). \tag{13}$$

By definition,

$$\begin{aligned}
&V_{\overline{p},n}(\delta_\Delta) = \int_0^\Delta k_n \alpha_n(\overline{p}(s)) d\Psi^i(s) + (N-n)\int_0^\Delta V_{\overline{p}^i(s),n+1} d\Psi^{-i}(s) + \\
&(1 - \sum_j \Psi^j(\Delta))[k_n \alpha_n(\overline{p}(\Delta)) - \beta(1 - \overline{p}(\Delta))],
\end{aligned}$$

where Ψ is the first-report distribution associated with the strategy profile where i plays δ_Δ and all $j \neq i$ play $F_{p,n}$. Meanwhile,

$$\begin{aligned}
&V_{\overline{p},n}(\delta_0) = k_n \alpha_n(\overline{p}) - \beta(1-\overline{p}) \\
&= \int_0^\Delta k_n \alpha_n(\overline{p}) d\Psi^i(s) + (N-n)\int_0^\Delta k_n \alpha_n(\overline{p}) - \beta(1 - \overline{p}^i(s)) d\Psi^{-i}(s) \\
&+ (1 - \sum_j \Psi^j(\Delta))(k_n \alpha_n(\overline{p}) - \beta(1 - \overline{p}(\Delta)).
\end{aligned}$$

Thus, in order for (13), for some $r \in (0, \overline{s})$,

$$k_n \alpha_n(\overline{p}) - \beta(1 - \overline{p}^i(r)) < V_{\overline{p}^i(r),n+1}. \tag{14}$$

First, consider the case where $\alpha_{n+1}(\overline{p}^i(r))) < 1$. Then, for $\varepsilon > 0$ sufficiently small

$$V_{\overline{p}^i(r),n+1} = V_{\overline{p}^i(r),n+1}(\delta_0) = k_{n+1}\alpha_{n+1}(\overline{p}^i(r)) - \beta(1-\overline{p}^i(r)) < k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}^i(r)),$$

where the first equality follows from Lemma 2. Thus, equation (14) is violated. Contradiction.

Next, consider the case where $\alpha_{n+1}(\overline{p}^i(r)) = 1$ and $\beta < k_n$. By the inductive assumption, it follows that $\alpha_{n+1}(q) = 1$ for all $q \leq \overline{p}^i(s)$. Thus, $F_{\overline{p}^i(s),n+1} = \delta_\infty$. So, we have that for ε sufficiently small:

$$\begin{aligned}&V_{\overline{p}^i(r),n+1} = V_{\overline{p}^i(r),n+1}(\delta_\infty) = \frac{k_{n+1}\overline{p}^i(r)}{N-n} \leq \overline{p}^i(r)k_n\alpha_n(\overline{p}) + (1-\overline{p}^i(r))k_n\alpha_n(\overline{p}) - \beta)\\ &= k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}^i(r)).\end{aligned}$$

Again, this is a contradiction of (14).

Finally, consider the case where $\alpha_{n+1}(\overline{p}^i(r)) = 1$ and $\beta \geq k_n$. Recall that $\alpha_n(q) = 1$ for all $q \geq p_n^*$. Thus, because $\alpha_n(\overline{p}) < 1$, it follows from (4) that $\alpha_n(\overline{p}(s))$ must be strictly increasing in s for some $s > r$. Formally, let

$$r' \equiv \inf\{s > r | \alpha_n(\overline{p}(s)) \text{ is strictly increasing}\}.$$

First, I claim that

$$k_n\alpha_n(\overline{p}(r')) - \beta(1-\overline{p}^i(r')) < V_{\overline{p}^i(r'),n+1}. \tag{15}$$

By the inductive assumption, since $\alpha_{n+1}(\overline{p}^i(r)) = 1$, it must be that $\alpha_{n+1}(q) = 1$ for all $q < \overline{p}^i(r)$. Because $\alpha_n(\overline{p}(s))$ is weakly decreasing in s for $s \in [r, r']$, it follows by definition of $\overline{p}^i(s)$ that $\overline{p}^i(s) < \overline{p}^i(r)$ for all $s \in [r, r']$. Thus, for all $s \in [r, r']$

$$V_{\overline{p}^i(s),n+1} = \frac{k_{n+1}\overline{p}^i(s)}{N-n}.$$

Then, for all $s \geq r$,

$$\begin{aligned}k_n\alpha_n(\overline{p}(s)) - \beta(1-\overline{p}^i(s)) &< V_{\overline{p}^i(s),n+1}\\ \Leftrightarrow k_n\alpha_n(\overline{p}(s)) - \beta(1-\overline{p}^i(s)) &< \frac{k_{n+1}\overline{p}^i(s)}{N-n}\\ \Leftrightarrow \overline{p}^i(s) &< \frac{\beta - k_n\alpha_n(\overline{p}(s))}{\beta - k_{n+1}/(N-n)}.\end{aligned}$$

Now, because $\alpha_n(\overline{p}(s))$ is strictly decreasing on $s \in [0, r]$,

$$k_n\alpha_n(\overline{p}(r)) - \beta(1 - \overline{p}^i(r)) < k_n\alpha_n(\overline{p}) - \beta(1 - \overline{p}^i(r)) < V_{\overline{p}^i(r),n+1}.$$

where the second inequality holds for the same reason as (14). Thus we have

$$\overline{p}^i(r') < \overline{p}^i(r) < \frac{\beta - k_n\alpha_{n+1}(\overline{p}(r))}{\beta - k_{n+1}/(N-n)} < \frac{\beta - k_n\alpha_{n+1}(\overline{p}(r'))}{\beta - k_{n+1}/(N-n)},$$

which implies (15).

It follows from this that there exists an $r'' > r'$ such that for all $s \in [r', r'']$, $\alpha_n(\overline{p}(s))$ is weakly decreasing and $V_{\overline{p}^i(s),n+1} > k_n\alpha_n(\overline{p}(r')) - \beta(1 - p^i(s))$. I now claim that

$$V_{\overline{p}(r'),n}(\delta_0) < V_{\overline{p}(r'),n}(\delta_{r''-r'}).$$

To see why, note that by definition,

$$\begin{aligned} V_{\overline{p}(r'),n}(\delta_{r''-r'}) - V_{\overline{p}(r'),n}(\delta_0) = \int_{r'}^{r''} k_n[\alpha_n(p(s)) - \alpha_n(p(r'))]d\Psi^i(s) + \\ \int_{r'}^{r''} [V_{p^i(s),n+1} - (k_n\alpha_n(p(r')) - \beta(1 - p^i(s)))]d\Psi^{-i}(s) \\ + \sum_j (\Psi^j(r'') - \Psi^j(r'))k_n(\alpha_n(p(r'')) - k_n\alpha_n(p(r'))). \end{aligned}$$

Since $\alpha_n(p(s)) \geq \alpha_n(p(r'))$ and $V_{p^i(s),n+1} > k_n\alpha_n(p(r')) - \beta(1 - p^i(s))$ for all $s \in [r', r'']$, it follows that $V_{\overline{p}(r'),n}(\delta_{r''-r'}) - V_{\overline{p}(r'),n}(\delta_0) > 0$. This contradicts Lemma 2. $\square$

Proof of Proposition 2. Proof by induction. Fix an n, and assume that $\alpha_m(p)$ satisfies the above for all $m > n$ such that (p, m) is on-path.

I begin by showing that (ODE) must hold whenever $\alpha_n(p) < 1$. To this end, assume that $\alpha_n(p) < 1$. By Lemma 2, there exists an $\varepsilon > 0$ such that for all $\Delta \in (0, \varepsilon)$,

$$\frac{V_{p,n}(\delta_\Delta) - V_{p,n}(\delta_0)}{\Delta} = 0. \tag{16}$$

By definition,

$$V_{p,n}(\delta_0) = k_n\alpha_n(p) - \beta(1 - p).$$

Meanwhile,

$$V_{p,n}(\delta_\Delta) = \int_0^\Delta k_n \alpha_n(p(s)) d\Psi^i(s) + (N-n) \int_0^\Delta V_{p^{-i}(s),n+1} d\Psi^{-i}(s) + \\ (1 - \sum_j \lim_{s \to \Delta-} \Psi^j(s))[k_n \alpha_n(p(\Delta)) - \beta(1-p(\Delta))],$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play the equilibrium strategy $F_{p,n}$. Specifically, for all $s > 0$,

$$\Psi^i(s) = p\lambda \int_0^s e^{-\lambda r(N-n+1)}(1 - F_{p,n}(r))^{N-n} dr$$

$$\Psi^{-i}(s) = p \int_0^s e^{-\lambda r(N-n)}(1-F_{p,n}(r))^{N-n-1} d(-e^{-\lambda r}(1-F_{p,n}(r))) + (1-p) \int_0^s (1-F_{p,n}(r))^{N-n-1} dF_{p,n}(r).$$

It follows from Lemma 1 that, for all j, Ψ^j is absolutely continuous on $[0, \Delta)$, i.e.,

$$\Psi^j(s) = \int_0^s \psi^j(r) dr.$$

where ψ^i and ψ^{-i} are given by the following:

$$\psi^i(r) = p\lambda e^{-\lambda r(N-n+1)}(1 - F_{p,n}(r))^{N-n}$$

$$\psi^{-i}(s) = pe^{-\lambda s(N-n+1)}(\lambda + F'_{p,n}(s+) - \lambda F_{p,n}(s))(1-F_{p,n}(s))^{N-n-1} + (1-p)(1-F_{p,n}(s))^{N-n-1} F'_{p,n}(s+).$$

Substituting the expressions for both $V_{p,n}(\delta_0)$ and $V_{p,n}(\delta_\Delta)$ into (16) and rearranging, we obtain that for all $\Delta \in (0, \varepsilon)$,

$$K_1(\Delta) + K_2(\Delta) + K_3(\Delta) = 0 \tag{17}$$

where

$$K_1(\Delta) \equiv \frac{\int_0^\Delta k_n[(\alpha_n(p(s)) - \alpha_n(p)) + \beta(1-p)]\psi^i(s) ds}{\Delta}$$

$$K_2(\Delta) \equiv \frac{(N-n) \int_0^\Delta [V_{p^{-i}(s),n+1} - k_n \alpha_n(p) + \beta(1-p)]\psi^{-i}(s) ds}{\Delta}$$

$$K_3(\Delta) \equiv \frac{(1 - \sum_j \lim_{s \to \Delta-} \Psi^j(\Delta))[k_n(\alpha_n(p(\Delta)) - \alpha_n(p)) + \beta(p(\Delta) - p)]}{\Delta}.$$

Now, we consider $\lim_{\Delta \to 0+}$ of $K_1(\Delta)$, $K_2(\Delta)$, and $K_3(\Delta)$ separately.

For $K_1(\Delta)$, it follows from L'Hôpital's Rule, together with the continuity of $\alpha_n(p(\Delta))$

(Lemma 4) and $\psi^i(\Delta)$ in Δ that

$$\lim_{\Delta\to 0+} K_1(\Delta) = \lim_{\Delta\to 0+}[k_n(\alpha_n(p(\Delta)) - \alpha_n(p)) + \beta(1-p)]\psi^i(\Delta) = \beta(1-p)\psi^i(0) = \beta(1-p)p\lambda.$$

For $K_2(\Delta)$, it again follows from L'Hôpital's Rule, together with the right-continuity of $V_{p^{-i}(\Delta),n+1}$ in Δ that

$$\begin{aligned}\lim_{\Delta\to 0+} K_2(\Delta) &= (N-n)\lim_{\Delta\to 0+}[V_{p^{-i}(\Delta),n+1} - k_n\alpha_n(p) + \beta(1-p)]\psi^{-i}(\Delta)\\ &= (N-n)[V_{p^{-i},n+1} - k_n\alpha_n(p) + \beta(1-p)](\frac{\lambda p}{\alpha_n(p)}),\end{aligned}$$

where the final inequality follows from the fact that at all (p,n) on-path, $\alpha_n(p) = \frac{\lambda p}{\lambda p + F'_{p,n}(0)}$.

For $K_3(\Delta)$, first note that by the continuous differentiability of $\Psi^j(s)$ that

$$\lim_{\Delta\to 0+}\sum_j \lim_{s\to\Delta-}\Psi^j(s) = 0.$$

Thus, it follows from the right-differentiability of $\alpha_n(p(\Delta))$ in Δ that

$$\begin{aligned}\lim_{\Delta\to 0+} K_3(\Delta) = k_n\lim_{\Delta\to 0}\frac{\alpha_n(p(\Delta))-\alpha_n(p)}{\Delta} + \beta\lim_{\Delta\to 0+}\frac{p(\Delta)-p}{\Delta} &= k_n\frac{d}{d\Delta}\alpha_n(p(\Delta))\Big|_{\Delta=0+} + \beta p'(\Delta)\Big|_{\Delta=0+}\\ = p'(\Delta)\Big|_{\Delta=0+}[k_n\alpha'_n(p)+\beta] &= -\lambda p(N-n+1)(1-p)[k_n\alpha'_n(p)+\beta].\end{aligned}$$

Since we have shown that $\lim_{\Delta\to 0+} K_1(\Delta)$, $\lim_{\Delta\to 0+} K_2(\Delta)$, and $\lim_{\Delta\to 0+} K_3(\Delta)$ exist, and are given by the above expressions, it follows from (17) that

$$\lim_{\Delta\to 0+} K_1(\Delta) + \lim_{\Delta\to 0+} K_2(\Delta) + \lim_{\Delta\to 0+} K_3(\Delta) = 0.$$

Substituting the above expressions for $K_1(\Delta)$, $K_2(\Delta)$ and $K_3(\Delta)$, we obtain (ODE).

Now, we wish to establish that (ODE) must hold whenever $k_n \geq \beta$ or $p > p_n^*$. It follows from Proposition 1 that $\alpha_n(p) < 1$, and thus by the above, (ODE) must hold.

Finally, we establish the two limit conditions presented in the proposition. We begin by establishing that when $k_n \geq \beta$, $\lim_{p\to 0+}\alpha_n(p) = \beta/k_n$. To this end, first note by Lemma 2 that for all $p > 0$, $V_{p,n}(\delta_0) = V_{p,n}(\delta_\infty)$. Note further that

$$\lim_{p\to 0+} V_{p,n}(\delta_\infty) = 0.$$

Thus,

$$\lim_{p\to 0+} V_{p,n}(\delta_0) = \lim_{p\to 0+} k_n\alpha_n(p) - \beta = 0,$$

and therefore, $\lim_{p\to 0+}\alpha_n(p) = \frac{\beta}{k_n}$. Next, let us consider the case where $k_n < \beta$. That $\lim_{p\to p_n^*+}\alpha_n(p) = 1$ follows from Lemma 4, since by Proposition 1, $\alpha_n(p_n^*) = 1$. □

Before proceeding with the rest of the characterization, I define a problem (P) on α. I then show that α consistutes an equlibrium if and only if it satisfies (P) (Lemma 5). Thus, existence and uniqueness of an equilibrium (Theorem 1) will reduce to establishing a unique solution to (P).

Definition 3. α is a solution to (P) if it satisfies the following for all $n \leq N$ and $p \in (0, 1]$:

1. If $k_n < \beta$ and $p \leq p_n^* \equiv \frac{k_n-\beta}{k_n/n-\beta}$, then $\alpha_n(p) = 1$.
2. If $k_n \geq \beta$ or $p < p_n^*$, then α satisfies (ODE), with limit condition $\lim_{p\to 0+}\alpha_n(p) = \beta/k_n$ if $k_n \geq \beta$ and $\lim_{p\to p_n*+}\alpha_n(p) = 1$ if $k_n < \beta$.
3. $\alpha_n(1) = 0$.

Lemma 5. *(α, F) is an equilibrium if and only if at all (p, n) on-path, α is both consistent with F and a solution to (P).*

Proof of Lemma 5. Fix an (α, F). I begin by establishing the necessity of the three conditions specified in Definition 3 for (α, F) to be an equilibrium. First we establish the necessity of part 3. of Definition 3. To this end, recall that by the selection assumption, $F_{1,n}(0) = 1$. Thus, it follows from (4) that $\alpha_n(1) = 0$ if $(p = 1, n)$ is on-path. Parts 1. and 2. of Definition (3) follow immediately from Proposition 1 and Proposition 2, respectively.

Next, we establish the sufficiency of the above conditions for (α, F) to be an equilibrium. We begin by considering the case in which $k_n < \beta$ and $p \leq p_n^*$. It follows from Definition 3 that $\alpha_n(q) = 1$ for all $q \leq p$. Thus, by (4), $F_{p,n} = \delta_\infty$. We want to show that there exist no profitable deviations in this case, i.e., that $V_{p,n} = V_{p,n}(\delta_\infty)$. It suffices to show that

$$V_{p,n}(\delta_\infty) \geq V_{p,n}(\delta_s) \text{ for all } s \in [0, \infty). \tag{18}$$

First, note that for all $s \in (0, \infty)$,

$$V_{p,n}(\delta_s) = k_n(1 - p(1 - e^{-\lambda s(N-n+1)})(\frac{N-n}{N-n+1})) - \beta(1-p) \leq k_n - \beta(1-p) = V_{p,n}(\delta_0).$$

Further, $k_n \leq \beta$ and $p \leq p_n^*$ implies that

$$V_{p,n}(\delta_0) = k_n - \beta(1-p) \leq \frac{k_n}{N-n+1} = V_{p,n}(\delta_\infty).$$

Thus, $V_{p,n}(\delta_\infty) \geq V_{p,n}(\delta_s)$ for all $s \in [0, \infty)$.

Next, we show that $F_{p,n}$ is optimal when $k_n > \beta$ or $p < p_n^*$. We begin by showing that

$$\frac{d}{d\Delta} V_{p,n}(\delta_\Delta) = 0 \text{ for all } \Delta \in [0,\infty) \text{ if } k_n \geq \beta \text{ and for all } \Delta \in [0, t^*) \text{ if } k_n < \beta \tag{19}$$

where t^* is the unique solution to $p(t^*) = p_n^*$. Note that

$$\begin{aligned} V_{p,n}(\delta_\Delta) = \int_0^\Delta k_n \alpha_n(p(s)) d\Psi^i(s) + (N-n) \int_0^\Delta V_{p^i(s),n+1} d\Psi^{-i}(s) + \\ (1 - \sum_j \Psi^j(\Delta))(\alpha_n(p(\Delta)) - \beta(1-p(\Delta))), \end{aligned} \tag{20}$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p,n}$. Then,

$$\begin{aligned} &\frac{d}{d\Delta} V_{p,n}(\delta_\Delta) \\ &= k_n \alpha_n(p(\Delta)) \Psi^{i\prime}(\Delta) + (N-n) V_{p^i(\Delta),n+1} \Psi^{-i\prime}(\Delta) + (1 - \sum_j \Psi^j(\Delta)) p'(\Delta) [\alpha_n'(p(\Delta)) + \beta] \\ &- \sum_j \Psi^{j\prime}(\Delta)(k_n \alpha_n(p(\Delta)) - \beta(1-p(\Delta))) \\ &= (N-n)[V_{p^i(\Delta),n+1} - k_n \alpha_n(p(\Delta)) + \beta(1-p(\Delta))] \Psi^{-i\prime}(\Delta) - \beta(1-p(\Delta)) \Psi^{i\prime}(\Delta) \\ &+ (1 - \sum_j \Psi^j(\Delta)) p'(\Delta)(k_n \alpha_n'(p(\Delta)) + \beta), \end{aligned}$$

where $\Psi^{i\prime}(t) \equiv \frac{d}{dt}\Psi^i(t)$.

In the above, the existence of $\Psi^{j\prime}(\Delta)$ follows from the differentiability of α_n at $p(\Delta)$, and thus, the differentiability of $F_{p,n}$ at Δ. We wish to show that $\frac{d}{d\Delta} V_{p,n}(\delta_\Delta) = 0$. To this end, we begin by deriving expressions for $\Psi^{i\prime}(\Delta)$ and $\Psi^{-i\prime}(\Delta)$. First, it follows by definition of the first-report distribution that:

$$\Psi^i(\Delta) = p\lambda \int_0^\Delta (1 - F_{p,n}(s))^{N-n} e^{-\lambda(N-n+1)s} ds.$$

Differentiating this, we obtain:

$$\Psi^{i\prime}(\Delta) = p\lambda(1 - F_{p,n}(\Delta))^{N-n} e^{-\lambda(N-n+1)\Delta}.$$

Meanwhile:

$$\Psi^{-i}(\Delta) = p\int_0^\Delta (1-F_{p,n}(s))^{N-n-1} e^{-\lambda(N-n)s} d((F_{p,n}(s)-1)e^{-\lambda s}) + (1-p)\int_0^\Delta (1-F_{p,n}(s))^{N-n-1} F'_{p,n}(s)ds.$$

where the existence of $F'_{p,n}(s)$ again follows from the assumption that α_n is differentiable at $p(s)$. Differentiating this, we obtain:

$$\begin{aligned}\Psi^{-i\prime}(\Delta) &= p(1 - F_{p,n}(\Delta))^{N-n-1} e^{-\lambda\Delta(N-n+1)}[F'_{p,n}(\Delta) + \lambda(1 - F_{p,n}(\Delta))] + (1-p)(1 - F_{p,n}(\Delta))^{N-n-1} F'_{p,n}(\Delta) \\ &= (1 - F_{p,n}(\Delta))^{N-n}[\frac{F'_{p,n}(\Delta)}{1 - F_{p,n}(\Delta)}(pe^{-\lambda\Delta(N-n+1)} + (1-p)) + pe^{-\lambda\Delta(N-n+1)}\lambda].\end{aligned}$$

It follows from (4) and (3) that

$$\frac{F'_{p,n}(\Delta)}{1 - F_{p,n}(\Delta)} = \lambda p(\Delta)(\frac{1}{\alpha_n(p(\Delta))} - 1).$$

Substituting this, along with the definition of $p(\Delta)$, we obtain:

$$\Psi^{-i\prime}(\Delta) = \lambda(1 - F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta(N-n+1)} + (1-p))\frac{p(\Delta)}{\alpha_n(p(\Delta))}.$$

Note further that

$$1 - \sum_j \Psi^j(\Delta) = (1 - F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta(N-n+1)} + (1-p)). \tag{21}$$

Substituting the expressions for $\Psi^{i\prime}(\Delta)$, $\Psi^{-i\prime}(\Delta)$, and $1 - \sum_j \Psi^j(\Delta)$ into the expression for $\frac{d}{d\Delta}V_{p,n}(\delta_\Delta)$ we obtain:

$$\begin{aligned}\frac{d}{d\Delta}V_{p,n}(\delta_\Delta) = K[&\frac{(N-n)}{\alpha_n(p(\Delta))}(V^i_{p(\Delta),n+1} - k_n\alpha_n(p(\Delta)) + \beta(1-p(\Delta))(1-\alpha_n(p(\Delta)))) \\ &- k_n\alpha'_n(p(\Delta))(1-p(\Delta))(N-n+1)].\end{aligned}$$

where $K \equiv \lambda(1 - F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta(N-n+1)} + (1-p))p(\Delta)$. Because (ODE) is satisfied at $(p(\Delta), n)$, using it to substitute in for $\alpha'_n(p(\Delta)$, we obtain (19).

Now, consider the case where $k_n \geq \beta$. To show $F_{p,n}$ is optimal, it suffices to show that all pure strategies δ_Δ yield the same payoff, i.e., that

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_\Delta) \tag{22}$$

for all $\Delta \in [0, \infty]$. It follows from (19) that (22) holds for all $\Delta \in [0, \infty)$. It remains to show that (22) holds for $\Delta = \infty$. By (19),

$$\begin{aligned}
&V_{p,n}(\delta_0) = \lim_{\Delta\to\infty} V_{p,n}(\delta_\Delta) \\
&= \lim_{\Delta\to\infty} \int_0^\Delta k_n\alpha_n(p(s))d\Psi^i(s) + (N-n)\lim_{\Delta\to\infty}\int_0^\Delta V_{p^i(s),n+1}d\Psi^{-i}(s) + \\
&\lim_{\Delta\to\infty}(1-\sum_j \Psi^j(\Delta))(k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta))) \\
&= \int_0^\infty k_n\alpha_n(p(\Delta))d\Psi^i(s) + (N-n)\int_0^\infty V_{p^i(\Delta),n+1}d\Psi^{-i}(s) = V_{p,n}(\delta_\infty),
\end{aligned}$$

where the third equality follows from the limit condition $\lim_{p\to 0+}\alpha_n(p) = \beta/k_n$:

$$\lim_{\Delta\to\infty} k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta)) = \lim_{p\to 0+} k_n\alpha_n(p) - \beta = 0.$$

Finally, consider the case where $k_n < \beta$ and $p > p_n^*$. Because $\alpha_n(p(s)) = 1$ for all $s > t^*$, by (4), it follows that $F'_{p,n}(s) = 0$ for all $s > t^*$. Thus, the support of $F_{p,n}$ is a subset of $[0, t^*] \cup \infty$. Thus, to show $F_{p,n}$ is optimal, it suffices to show that δ_Δ is optimal for $\Delta \in [0, t^*] \cup \infty$. I first show that

$$V_{p,n}(\delta_\Delta) = V_{p,n}(\delta_0) \text{ for all } \Delta \in [0, t^*] \cup \infty \tag{23}$$

and then show

$$V_{p,n}(\delta_{t^*}) \geq V_{p,n}(\delta_\Delta)\text{for all } \Delta \in (t^*, \infty). \tag{24}$$

To show (23), recall that it follows from (19) that

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_\Delta) \text{ for all } \Delta \in [0, t^*).$$

It remains to show $V_{p,n}(\delta_0) = V_{p,n}(\delta_s)$ for $s \in \{t^*, \infty\}$. For $s = t^*$, it follows from the above that

$$V_{p,n}(\delta_0) = \lim_{\Delta\to t^*-} V_{p,n}(\delta_\Delta) = V_{p,n}(\delta_{t^*}),$$

where the final inequality follows from (20), and the continuity of $\alpha_n(p(t))$ and Ψ^j at t^*. I will now show $V_{p,n}(\delta_{t^*}) = V_{p,n}(\delta_\infty)$. Note that for all $\Delta \in [t^*, \infty]$:

$$V_{p,n}(\delta_\Delta) = \int_0^{t^*} k_n \alpha_n(p(s)) d\Psi^i(s) + (N-n) \int_0^{t^*} V_{p^i(s),n+1} d\Psi^{-i}(s) + (1 - \sum_j \Psi^j(t^*)) V_{p_n^*,n}(\delta_{\Delta - t^*}).$$

Thus, to show $V_{p,n}(\delta_{t^*}) = V_{p,n}(\delta_\infty)$, it suffices to show $V_{p_n^*,n}(\delta_0) = V_{p_n^*,n}(\delta_\infty)$. It follows from the definition of p_n^* that:

$$V_{p_n^*,n}(\delta_0) = k_n - \beta(1 - p_n^*) = \frac{k_n p_n^*}{n} = V_{p_n^*,n}(\delta_\infty).$$

Similarly, to show (24), it suffices to show that $V_{p_n^*,n}(\delta_0) \geq V_{p_n^*,n}(\delta_\Delta)$ for all $\Delta \in (0, \infty)$, which we have established in (18). □

Proof of Theorem 1. Fix an n. Assume by induction that there exists a unique solution to (P) for all $m > n$. We wish to show that there exists a unique solution to (P) for n. It suffices to show there exists a unique solution to the following two problems, when $\beta \leq k_n$ and $\beta > k_n$, respectively:

$$\text{(ODE) is satisfied on } (0,1) \text{, and } \lim_{p \to 0+} \alpha_n(p) = \beta / k_n \qquad \text{(LP: } \beta \leq k_n\text{)}$$

$$\text{(ODE) is satisfied on } (0,p^*) \text{, and } \lim_{p \to p_n^*+} \alpha_n(p) = 1. \qquad \text{(LP: } \beta > k_n\text{)}$$

To establish existence and uniqueness to the two above problems, we proceed by extending them to two boundary value problems. To this end, we begin by defining the problem (ODE′), which is identical to (ODE), except that it is well-defined when $p^i \geq 1$. Specifically, define:

$$\alpha_n'(p) = -\frac{1}{k_n(1-p)\alpha_n(p)} \frac{N-n}{N-n+1} [k_n \alpha_n(p) - \tilde{V}_{p^i,n+1} - \beta(1 - \alpha_n(p))(1-p)], \qquad \text{(ODE′)}$$

where

$$\tilde{V}_{p^i,n+1} = \begin{cases} V_{p^i,n+1} & \text{if } p^i \in (0,1) \\ 0 & \text{if } p^i \geq 1. \end{cases}$$

Now define two boundary value problems on (ODE′):

$$\text{(ODE′) is satisfied on } [0,1) \text{, and } \alpha_n(0) = \beta / k_n \qquad \text{(BVP: } \beta \leq k_n\text{)}$$

$$\text{(ODE}'\text{) is satisfied on } (0, p_n^*] \text{, and } \alpha_n(p^*) = 1. \qquad \text{(BVP: } \beta \geq k_n\text{)}$$

I claim that the existence and uniqueness of a solution to (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$) implies the existence and uniqueness of a solution to (LP: $\beta \leq k_n$) and (LP: $\beta > k_n$), respectively. First, consider the case where $k_n \geq \beta$. Assume that there exists a unique solution α_n to (BVP: $\beta \leq k_n$). Note that in order for α_n to satisfy (BVP: $\beta \leq k_n$), it must be that $\lim_{p\to 0+} \alpha_n(p) = k_n/\beta$. Furthermore, (ODE) and (ODE′) are equivalent on $(0, 1)$. It follows that α_n is a solution to (LP: $\beta \leq k_n$), thus establishing existence. To establish uniqueness, assume by contradiction there exists some $\tilde{\alpha}_n$ defined on $p \in (0, 1)$ that is a solution to (LP: $\beta \leq k_n$) where $\tilde{\alpha}_n(p) \neq \alpha_n(p)$. Now, define $\hat{\alpha}_n$, which extends the domain of $\tilde{\alpha}_n$, as follows:

$$\hat{\alpha}_n(p) = \begin{cases} \tilde{\alpha}_n(p) & \text{if } p \in (0,1) \\ k_n/\beta & \text{if } p = 0. \end{cases}$$

Because $\lim_{p\to 0+} \tilde{\alpha}_n(p) = k_n/\beta$, it follows that $\hat{\alpha}_n(p)$ satisfies (ODE′) on $p \in [0, 1]$ and is thus a solution to (BVP: $\beta \leq k_n$). Thus, (BVP: $\beta \leq k_n$) does not have a unique solution, a contradiction. The argument in the case where $k_n < \beta$ is analogous.

It remains to establish that there exist unique solutions to both (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$). We do this by invoking the Picard existence and uniqueness theorem, and thus begin by establishing that the right-hand side of (ODE′) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p for $p \in [-\varepsilon, 1)$ and $\alpha_n(p) \in [c, 1+\varepsilon]$ for any $c > 0$ and some $\varepsilon > 0$. Since $p^i \equiv \alpha_n(p) + (1 - \alpha_n(p))p$, it suffices to show that $\tilde{V}_{p^i,n+1}$ is Lipschitz continuous in p^i for $p^i \geq 0$. In the case where $n = 1$, $\tilde{V}_{p^i,n+1} = 0$ for all p^i, and this is immediate. Next, suppose $n > 1$. First, consider the case where $k_{n+1} \geq \beta$. It follows from Lemma 2 that:

$$\tilde{V}_{p^i,n+1} = \begin{cases} k_n\alpha_{n+1}(p^i) - \beta(1 - p^i) & \text{if } p^i < 1 \\ 0 & \text{if } p^i > 1. \end{cases}$$

Because $\tilde{V}_{p^i,n+1}$ is continuously differentiable in p^i when $p^i \neq 1$, to establish that it is Lipschitz continuous it suffices to show that $\lim_{p^i\to 1-} V_{p^i,n+1} = 0$. Suppose this does not hold, by contradiction. Because $\alpha_{n+1}(\cdot)$ satisfies (ODE), this implies that $\lim_{p^i\to 1-} \alpha'_{n+1}(p^i) = \infty$. This implies that $\lim_{p^i\to 1} \alpha_{n+1}(p^i) = \infty$, and thus that (ODE) is not satisfied at $p^i = 1$. Contradiction.

Next, consider the case where $k_{n+1} < \beta$. In this case:

$$\tilde{V}_{p^i,n+1} = \begin{cases} k_{n+1}p^i/(N-n) & \text{if } p^i < p^*_{n+1} \\ k_n\alpha_{n+1}(p^i) - \beta(1-p^i) & \text{if } p^i \in (p^*_{n+1}, 1) \\ 0 & \text{if } p^i = 1. \end{cases}$$

By the same reasoning as above, $\tilde{V}_{p^i,n+1}$ is Lipschitz continuous for all $p^i > p^*_{n+1}$. Furthermore, Lipschitz continuity holds for $p^i < p^*_{n+1}$. To show that Lipschitz continuity holds across all p^i, it suffices to show that $\tilde{V}_{\cdot,n+1}$ is differentiable at p^*_{n+1}. To this end, we take the left- and right- derivative of $\tilde{V}_{\cdot,n+1}$ at p^*_{n+1} and show that they are equal:

$$\frac{d}{dp}\tilde{V}_{p^*-,n+1} = \frac{k_{n+1}}{N-n}$$

$$\frac{d}{dp}\tilde{V}_{p^*+,n+1} = -k_{n+1}\alpha'_{n+1}(p^*_{n+1}) + \beta = \frac{k_{n+1}}{1-p^*_{n+1}}\frac{N-n-1}{N-n} + \beta = \frac{k_{n+1}}{N-n}.$$

Now, we show that there exists a unique solution for both (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$) in some neighborhood of their respective boundary conditions. By the Picard Theorem, this follows immediately from our above-established result that the right-hand side of (ODE) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p in some neighborhood of the boundary conditions $(\alpha_n(p) = 1, p = p^*_n)$ and $(\alpha_n(p) = \beta/k_n, p = 0)$.

Next, we seek to establish global existence and uniqueness of solutions to both (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$). First, consider (BVP: $\beta \geq k_n$). The argument for (BVP: $\beta \leq k_n$) follows analogously. Let $[p^*, \overline{p})$ denote the largest right-open interval such that existence and uniqueness are both satisfied. Assume by contradiction that $\overline{p} < 1$. Let $\alpha_n(p)$ denote the solution along this interval.

We begin by showing that on this interval, $\alpha_n(p) \in (\underline{\alpha}, 1]$, where $\underline{\alpha} > 0$ is some constant. The upper bound is established as follows: suppose by contradiction that $\alpha_n(p) > 1$ somewhere on the interval. By the continuous differentiability of α_n along the interval, there must exist some $q < p$ such that $\alpha_n(q) = 1$ and $\alpha'_n(q) \geq 0$. However, it follows from (ODE′) that

$$\alpha'_n(q) = -\frac{1}{k_n(1-q)}\frac{N-n}{N-n+1}[k_n - \tilde{V}_{p^i,n+1}] < 0,$$

where the strict inequality follows from the fact that $\tilde{V}_{p^i,n+1} \leq k_{n+1} < k_n$. Contradiction. The lower bound is established as follows: suppose by contradiction that such a lower bound does not exist. Then, again by the continuous differentiability of α_n along the interval, there exists some $\hat{p} \in [p_n^*, \overline{p})$ such that

$$\lim_{p \to \hat{p}-} \alpha_n(p) = 0 \text{ and } \alpha_n(p) > 0 \text{ for all } p < \hat{p}.$$

However, it then follows from (ODE) that $\lim_{p \to \hat{p}-} \alpha_n'(p) = \infty$. Thus, (ODE′) is not satisfied on $[p_n^*, \overline{p})$. Contradiction.

Having established that on $[p^*, \overline{p})$, $1 \geq \alpha_n(p) > \underline{\alpha} > 0$, it follows from (ODE′), and the observation that $\tilde{V}_{p^i,n+1}$ is bounded, that α_n' is also bounded on this range. Thus, it follows that $\lim_{p \to \overline{p}-} \alpha_n(p) \equiv \overline{\alpha} > 0$ exists.

Now, consider the following modified boundary value problem, which is identical to (BVP: $\beta \geq k_n$), except with boundary condition $(\overline{p}, \overline{\alpha})$. Recall we have shown that (ODE′) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p in some neighborhood of $(\hat{p}, \hat{\alpha})$. Thus, we can again apply the Picard Theorem to obtain that there exists a unique solution to the modified boundary value problem in some neighborhood of $(\overline{p}, \overline{\alpha})$. I.e., there exists some $\varepsilon > 0$ such that there is a unique solution $\tilde{\alpha}_n(p)$ on interval $(\overline{p} - \varepsilon, \overline{p} + \varepsilon)$. We now "paste" this solution $\tilde{\alpha}_n$, with our prior solution α_n. Let

$$\hat{\alpha}_n(p) = \begin{cases} \alpha_n(p) & \text{if } p \in [p_n^*, \overline{p}) \\ \tilde{\alpha}_n(p) & \text{if } p \in [\overline{p}, \overline{p} + \varepsilon). \end{cases}$$

Note that $\hat{\alpha}_n(p)$ is a unique solution to (BVP: $\beta \geq k_n$) on $[p_n^*, \overline{p} + \varepsilon)$, which contradicts our earlier assumption that $[p^*, \overline{p})$ was the largest right-open interval such that existence and uniqueness are satisfied. Contradiction. □

Proof of Proposition 3 and Proposition 4. Let us begin by showing that $\alpha_n(p)$ is weakly decreasing in p for all (p, n) on-path. By Lemma 5, it follows that when $k_N < \beta$, $\alpha_N(p) = 1$ for all p, and when $k_N \geq \beta$, $\alpha_N'(p) = 0$ for all p. Thus, $\alpha_N(p)$ is constant in p. Now, consider the case where $n < N$. Assume by induction that $\alpha_{n+1}(p)$ is weakly decreasing in p whenever $(p, n+1)$ is on path.

Assume by contradiction that there exists some $\overline{p}$ such that α_n is strictly increasing. Note that by Lemma 5, $\alpha_n'(p) = 0$ whenever $\beta \geq k_n$ and $p \geq p_n^*$. Thus it must be that $\beta < k_n$ or $\overline{p} > p_n^*$. In this case, (ODE) must be satisfied. Now define the function $X(p)$ as follows:

$$X(p) \equiv k_n \alpha_n(p) - \beta(1 - p^i) - V_{p^i,n+1}. \tag{25}$$

Note that whenever (ODE) is satisfied, the following holds:

$$\alpha_n'(p) > (=)0 \text{ if and only if } X(p) < (=)0. \tag{26}$$

Thus, $X(\overline{p}) < 0$. Now, I claim that there exists $\underline{p} < \overline{p}$ such that $\lim_{p\to\underline{p}+} X(p) \geq 0$. To establish this, first suppose $k_n \geq \beta$. In this case,

$$\lim_{p\to 0+} X(p) = k_n \lim_{p\to 0+} \alpha_n(p) - \beta(1 - \lim_{p\to 0+}) - \lim_{p\to 0+} V_{\alpha_n(p),n+1} = (k_n+\beta) \lim_{p\to 0+} \alpha_n(p) - \beta - \lim_{p\to 0+} V_{\alpha_n(p),n+1}. \tag{27}$$

When $\lim_{p\to 0+} \alpha_{n+1}(\alpha_n(p)) < 1$, it follows from Lemma 2 that

$$\begin{aligned}\lim_{p\to 0+} V_{\alpha_n(p),n+1} = \lim_{p\to 0+} V_{\alpha_n(p),n+1}(\delta_0) &= k_{n+1} \lim_{p\to 0+} \alpha_{n+1}(\alpha_n(p)) - \beta(1 - \lim_{p\to 0+} \alpha_n(p)) \\ &= k_{n+1}\alpha_{n+1}(\beta/k_n) - \beta(1 - \beta/k_n).\end{aligned}$$

Because $k_n \geq \beta$, the final equality follows from Lemma 5. Substituting this into (27), we obtain

$$\lim_{p\to 0+} X(p) = \beta - k_{n+1}\alpha_{n+1}(\beta/k_n).$$

In the case where $k_{n+1} < \beta$, it follows directly that $\lim_{p\to 0+} X(p) \geq 0$. Otherwise, if $k_{n+1} \geq \beta$, then because $\lim_{p\to 0+} \alpha_{n+1}(p) = \beta/k_{n+1}$, it follows from the inductive assumption that $\alpha_{n+1}(p) \leq \beta/k_{n+1}$ for all p, and thus that $\lim_{p\to 0+} X(p) \geq 0$.

When $\lim_{p\to 0+} \alpha_{n+1}(\alpha_n(p)) = 1$, it follows from the inductive assumption that $\alpha_{n+1}(q) = 1$ for all $q \geq \lim_{p\to 0+} \alpha_n(p)$. Thus,

$$\lim_{p\to 0+} V_{p^i,n+1} = \lim_{p\to 0+} V_{p^i,n+1}(\delta_\infty) = \frac{k_{n+1}}{N-n}\frac{\beta}{k_n}.$$

Substituting into the above expression for $\lim_{p\to 0+} X(p)$, we obtain

$$\lim_{p\to 0+} X(p) = (\beta/k_n)(\beta - k_{n+1}/(N-n)) \geq 0,$$

where the inequality follows from the fact that $\alpha_{n+1}(\beta/k_n) = 1$, implying by Lemma 5 that $k_{n+1} \geq \beta$.

Next, consider the case where $k_n < \beta$. In this case,

$$\lim_{p\to p_n^*+} X(p) = k_n - \lim_{p^i\to 1-} V_{p^i,n+1}.$$

If $\lim_{p^i \to 1-} \alpha_{n+1}(p^i) < 1$, then by Lemma 2,

$$\lim_{p^i \to 1-} V_{p^i,n+1} = \lim_{p^i \to 1-} V_{p^i,n+1}(\delta_0) = k_{n+1} \lim_{p^i \to 1-} \alpha_{n+1}(p^i) < k_n.$$

Thus, in this case, we obtain that $\lim_{p \to p_n^* +} X(p) > 0$. Meanwhile, if $\lim_{p^i \to 1-} \alpha_{n+1}(p^i) = 1$, by the inductive assumption, $\alpha_{n+1}(p) = 1$ for all p. Thus,

$$\lim_{p^i \to 1-} V_{p^i,n+1} = \lim_{p^i \to 1-} V_{p^i,n+1}(\delta_\infty) = \frac{k_{n+1}}{N-n}.$$

So in this case as well, $\lim_{p \to p_n^* +} X(p) > 0$. We have thus shown that there always exists $\underline{p} < \overline{p}$ such that $\lim_{p \to \underline{p}+} X(p) \geq 0$.

Because $X(\overline{p}) < 0$, there must exist some $q \in [\underline{p} such that \overline{p}]$ $X(q) < 0$ and $X'(q) < 0$. Note that differentiating X, we obtain

$$X'(q) = k_n \alpha_n'(q) + \beta((1-q)\alpha_n'(q) + (1 - \alpha_n(q))) - \frac{d}{dq} V_{q^i,n+1}. \tag{28}$$

First, consider the case where $\alpha_{n+1}(q^i) < 1$. By Lemma 2,

$$V_{q^i,n+1} = V_{q^i,n+1}(\delta_0) = k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i). \tag{29}$$

Substituting this into (28), we obtain

$$X'(q) = k_n \alpha_n'(q) - k_{n+1}\alpha_{n+1}'(q^i)((1-q)\alpha_n'(q) + (1 - \alpha_n(q))).$$

Note that because $X(q) < 0$ it follows from (26) that $\alpha_n'(q) > 0$. Furthermore, by the inductive assumption, $\alpha_{n+1}'(q^i) \leq 0$. Thus, $X'(q) > 0$. Contradiction.

Next, consider the case where $\alpha_{n+1}(q^i) = 1$. By the inductive assumption, $\alpha_{n+1}(p) = 1$ for all $p \leq q^i$. Thus,

$$V_{q^i,n+1} = V_{q^i,n+1}(\delta_\infty) = \frac{k_{n+1} q^i}{N-n}.$$

Substituting this into (28), we obtain

$$X'(q) = k_n \alpha_n'(q) + (\beta - \frac{k_{n+1}}{N-n})((1-q)\alpha_n'(q) + (1 - \alpha_n(q))). \tag{30}$$

Because $\alpha_{n+1}(q^i) = 1$, by Lemma 5, it must be that $\beta \geq k_{n+1}$. Thus, $X'(q) > 0$. Contradiction.

Next, we show that if $k_N \geq \beta$, then $\alpha_n(p) = \beta / k_n$. Assume that $k_N \geq \beta$. First consider

$n = N$. By Lemma 5, $\alpha_n'(p) = 0$ for all p on-path, and thus, $\alpha_N(p)$ is constant in p. Since Lemma 5 also states that $\lim_{p\to 0+} k_N \alpha_N(p) = \beta$, it must be that $\alpha_N(p) = \beta/k_N$ for all p. Now, consider $n < N$. Assume by induction that $\alpha_{n+1}(p) = \beta/k_{n+1}$ for all p. We begin by showing that $\alpha_n(p)$ is constant in p. Since $k_n \geq \beta$, by Lemma 5, (ODE) must hold at all p. By (26), showing $\alpha_n(p)$ is constant in p is equivalent to showing that $X(p) = 0$. To establish this, I begin by claiming that $V_{p^i,n+1} = V_{p^i,n+1}(\delta_0)$. In the case where $k_{n+1} > \beta$, it follows from Proposition 1 that $\alpha_{n+1}(p^i) < 1$, and thus this follows from Lemma 2. In the case where $k_{n+1} = \beta$, it follows that $k_m = \beta$ for all $m \geq n+1$, and by Proposition 1, $\alpha_m(p) = 1$ for all p. Thus, $V_{p,n+1}(\delta_s) = p\beta$. for all $\delta \in [0, \infty]$ and all p. Thus, $V_{p^i,n+1} = V_{p^i,n+1}(\delta_0)$. Having established that $V_{p^i,n+1} = V_{p^i,n+1}(\delta_0)$, we have:

$$V_{p^i,n+1} = k_{n+1}\alpha_{n+1}(p^i) - \beta(1-p^i) = \beta p^i.$$

Substituting this into (25), we obtain $X(p) = k_n\alpha_n(p) - \beta$. Since we established above that $\alpha_n(p)$ is weakly decreasing, $\alpha_n(p) \leq k_n/\beta$ for all p, and thus $X(p) \leq 0$. Separately, by (26) $\alpha_n(p)$ weakly decreasing implies that $X(p) \geq 0$. Combining these inequalities, we have $X(p) = 0$.

Finally, I show that $k_N < \beta$ implies that $\alpha_n'(p) < 0$ whenever $\alpha_n(p) < 1$. Suppose $k_N < \beta$, and suppose by contradiction that at some q such that $\alpha_n(q) < 1$, $\alpha_n'(q) = 0$. It follows from (26) that $X(q) = 0$.

First, suppose $\alpha_{n+1}(q^i) = 1$. Recall from (30) that

$$X'(q) = k_n\alpha_n'(q) + (\beta - \frac{k_{n+1}}{N-n})((1-q)\alpha_n'(q) + (1-\alpha_n(q)) = (\beta - \frac{k_{n+1}}{N-n})(1-\alpha_n(q)).$$

Now, I claim that $\beta > \frac{k_{n+1}}{N-n}$. When $n = N-1$, this follows directly from the assumption that $k_N < \beta$. Meanwhile, when $n < N-1$, because $\alpha_{n+1}(q^i) = 1$, this is a result of Proposition 1. Thus, $X'(q) > 0$. Since $X(q) = 0$, for some $p < q$, we must have $X(p) < 0$. By (28), $\alpha_n'(p) > 0$. This contradicts the above-established assertion that $\alpha_n(p)$ is weakly decreasing in p.

Next, suppose $\alpha_{n+1}(q^i) < 1$. By (29), in this case:

$$X'(q) = k_n\alpha_n'(q) - k_{n+1}\alpha_{n+1}'(q^i)[(1-q)\alpha_n'(q) + (1-\alpha_n(q))] = -k_{n+1}\alpha_{n+1}'(q)[1-\alpha_n(q)] > 0.$$

Again, this implies that there exists some $p < q$ such that $X(p) < 0$ and thus that $\alpha'(p) > 0$. Contradiction. $\square$

Proof of Corollary 3. It suffices to show that

$$\lim_{p \to 0+} b_{n+1}(\tilde{p}) - b_n(p) > 0.$$

To this end, note that it follows from Proposition 2 and Equation 4 that $\lim_{p \to 0+} b_n(p) = 0$. Next, note that

$$\lim_{p \to 0+} \tilde{p} = \lim_{p \to 0+} \alpha_n(p) = \beta / k_n,$$

where the final equality follows from Proposition 2. Thus,

$$\lim_{p \to 0+} b_{n+1}(\tilde{p}) = b_{n+1}(\lim_{p \to 0+} \tilde{p}) = b_{n+1}(\beta / k_n),$$

where the first equality follows from the continuity of b_{n+1} in p. Thus,

$$\lim_{p \to 0+} [b_{n+1}(\tilde{p}) - b_n(p)] = b_{n+1}(\beta / k_n) > 0.$$

$\square$

Proof of Corollary 4. It suffices to show that

$$\lim_{p \to 0+} \alpha_1(p) - \alpha_1^m(p) > 0.$$

First, by Proposition 2, $\alpha_1^m(p) = \beta / k^m$. Next, since $k_N < \beta$, $\lim_{p \to 0+} \alpha_1(p) = \beta / k_1$. Thus,

$$\lim_{p \to 0+} \alpha_1(p) - \alpha_1^m(p) = \beta / k_1 - \beta / k_m > 0,$$

where the final inequality follows from teh fact that $k_2 > 0$, and thus $k^m > k_1$. $\square$

Appendix D Commitment solution

Here, we seek the optimal solution to the monopoly case of the baseline model in which the firm has the ability to commit to a reporting strategy. The only modification we introduce is that rather than F and α being determined simultaneously as they are in equilibrium, the firm chooses its strategy F before α is determined. Thus, in the commitment case, the credibility function is a function of the firm's strategy. We formalize this dependence by denoting the firm's credibility function as α_F. α_F is then given by (4) as in the non-commitment case, except that the strategy F upon which it is computed is the firm's choice of strategy, rather than the equilibrium strategy.

Because we are considering the monopoly case only, I will be dropping the n index from all functions and expressions. Furthermore, for convenience, I will be writing all functions as a function of calendar time t, rather than the common belief p as in the baseline model. Writing the functions in this way is without loss, since under a monopoly there is a one-to-one correspondence between the calendar time t and the common belief p.

The firm's objective is to choose a permissible strategy $F \in \mathcal{F}$ which maximizes its expected payoff over the course of the game. Specifically, its problem is given by the following:

$$\max_{F \in \mathcal{F}} \int_0^\infty [\alpha_F(t) - \beta(1 - p(t))(1 - \alpha_F(t))] d\Psi(t), \tag{31}$$

where, as in the baseline setup, $\Psi(t)$ denotes probability that the firm reports before time t under strategy F. Before proceeding, we highlight that the only difference between this problem and the problem of the monopoly case of the baseline model is that the credibility function is not taken as given, but is rather a function of the firm's choice of strategy F.

In the analysis that follows, it will be useful for us to cast this problem as a choice of an optimal credibility function α, rather than an optimal strategy F. To this end, I begin with a useful observation, which is analogous to Lemma 1, except under the commitment setting:

Lemma 6. *F must be continuous in equilibrium.*

We omit a proof for this claim, as it follows analogously to the proof for Lemma 1: if F exhibits a discontinuity at some time t, reporting at this time must yield a negative expected payoff. Thus, the firm can profitably deviate by shifting the mass that it had placed on reporting t to δ_∞.

It follows immediately from Lemma 6 that in equilibrium, both the firm's strategy F and the corresponding commitment function, α_F, are defined by the right-hazard rate $b(t)$ of the firm's strategy. That is,

$$\alpha_F(t) = \frac{\lambda p(t)}{\lambda p(t) + b(t)}.$$

It further follows that Ψ is continuous and can thus be written as a function of α_F as follows:

$$\Psi(t) = 1 - e^{-\int_0^t (b(s) + p(s)\lambda) ds} = 1 - e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)} ds}.$$

Having written Ψ in terms of α_F, and noting that at any given t $\alpha_F(t)$ is a one-to-one function of $b(t)$, we can cast the optimization problem given by (31) as one over α_F:

$$\max_{\alpha_F} \int_0^\infty \lambda p(t)[1-\beta(1-p(t))(\frac{1}{\alpha_F(t)}-1)]e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}.$$

In the following claim, I show that the optimal strategy for the firm consists of always truth telling (i.e., $\alpha_F(t) = 1$ for all t). In the proof that follows, I let $V(t, \alpha_F)$ denote the firm's value at time t given that it has chosen α_F.

Proposition 7. *In equilibrium, $\alpha_F(t) = 1$ for all t.*

Proof. Assume not, by contradiction. Then there exists a t^* such that $\alpha_F(t^*) < 1$. It follows from Lemma 6, and the assumption that F is right-continuously differentiable, that α_F must be right-continuous. Thus, there must exist a $\hat{\alpha} < 1$ and $\varepsilon > 0$ such that $\alpha_F(t) < \hat{\alpha}$ for all $t \in [t^*, t^*+\varepsilon]$.

Note that for any α_F, including the equilibrium α_F, we can write the time-0 value as follows:

$$V(0,\alpha_F) = \int_0^{t^*+\varepsilon} \lambda p(t)[1-\beta(1-p(t))](\frac{1}{\alpha_F(t)}-1)e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}dt + e^{-\int_0^{t^*+\varepsilon} \frac{\lambda p(s)}{\alpha_F(s)}ds}V(t^*+\varepsilon, \alpha_F). \tag{32}$$

Now, consider the following deviation $\tilde{\alpha}_F$, which is identical to α_F, except that it is 1 on the interval $[t^*, t^*+\varepsilon]$:

$$\tilde{\alpha}_F(t) = \begin{cases} 1 & \text{if } t \in [t^*, t^*+\varepsilon] \\ \alpha_F(t) & \text{otherwise.} \end{cases}$$

It follows from (32) that

$$V(0,\alpha_F) = V(0,\tilde{\alpha_F}) + \int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1-\beta(1-p(t))(\frac{1}{\alpha_F(t)}-1)]e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}dt - \int_{t^*}^{t^*+\varepsilon} \lambda p(t)e^{-\int_0^t \lambda p(s)ds}dt$$
$$+(e^{-\int_{t^*}^{t^*+\varepsilon} \frac{\lambda p(s)}{\alpha_F(s)}ds} - e^{-\int_{t^*}^{t^*+\varepsilon} \lambda p(s)ds})V(t^*+\varepsilon, \alpha_F) \tag{33}$$

We note the following two inequalities:

$$\int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1-\beta(1-p(t))(\frac{1}{\alpha_F(t)}-1)]e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)}ds}dt \leq \int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1-\beta(1-p(t))(\frac{1}{\overline{\alpha}}-1)]e^{-\int_0^t \frac{\lambda p(s)}{\overline{\alpha}}ds}dt$$
$$< \int_{t^*}^{t^*+\varepsilon} \lambda p(t)e^{-\int_0^t \lambda p(s)ds}dt$$

$$e^{-\int_{t^*}^{t^*+\varepsilon} \frac{\lambda p(s)}{\alpha_F(s)} ds} - e^{-\int_{t^*}^{t^*+\varepsilon} \lambda p(s) ds} \leq e^{-\int_{t^*}^{t^*+\varepsilon} \frac{\lambda p(s)}{\tilde{\alpha}} ds} - e^{-\int_{t^*}^{t^*+\varepsilon} \lambda p(s) ds} < 0$$

These two inequalities combined with (33) yields

$$V(0, \alpha_F) < V(0, \tilde{\alpha}),$$

and thus, $\tilde{\alpha}_F$ serves as a profitable deviation. Contradiction. $\square$

Appendix E Proofs: comparative statics

Proof of Proposition 5. First, we establish part (a). Fix all other parameters and let $0 < \beta < \tilde{\beta}$. Let α and $\tilde{\alpha}$ denote the equilibrium credibility functions under β and $\tilde{\beta}$, respectively. Fix an n and assume inductively that the proposition holds for $n+1$ if $n < N$. Note that for any (p, n) and t, $p(t)$ will be the same under β and $\tilde{\beta}$. Thus to show the above claim, it suffices to show that for any p, $\alpha_n(p)$ is weakly increasing in β, and strictly so whenever $\alpha_n(p) < 1$.

We begin by showing that $\alpha_n(p) = 1$ implies that $\tilde{\alpha}_n(p) = 1$. First, consider the case where $n = N$. By Proposition 2, $\alpha_N(p) = 1$ implies that $k_N \leq \beta$. Thus, $k_N < \tilde{\beta}$, which by Proposition 1 implies that $\tilde{\alpha}_N(p) = 1$. Next, consider the case where $n < N$, and assume $\alpha_n(p) = 1$. By Proposition 1, this implies that $k_n < \beta$ and $p \leq p_n^* \equiv \frac{\beta - k_n}{\beta - k_n/n}$. Further note that

$$\tilde{p}_n^* \equiv \frac{\tilde{\beta} - k_n}{\tilde{\beta} - k_n/n} > \frac{\beta - k_n}{\beta - k_n/n} \equiv p_n^*.$$

Thus, $k_n < \tilde{\beta}$ and $p < \tilde{p}_n^*$, which by Proposition 1 implies $\tilde{\alpha}_n(p) = 1$.

Now, suppose that $\alpha_n(p) < 1$. We wish to show that $\tilde{\alpha}_n(p) > \alpha_n(p)$. Suppose by contradiction that $\tilde{\alpha}_n(p) \leq \alpha_n(p)$. It follows from Proposition 2 that if $k_n > \tilde{\beta}$,

$$\lim_{q \to 0+} \alpha_n(q) = \beta / k_n < \tilde{\beta} / k_n = \lim_{q \to 0+} \tilde{\alpha}_n(q).$$

Meanwhile, if $k_n \leq \tilde{\beta}$.

$$\lim_{q \to \tilde{p}_n^* +} \alpha_n(q) < 1 = \lim_{q \to \tilde{p}_n^* +} \tilde{\alpha}_n(q).$$

To see why the latter must must hold, first consider the case where $n = 1$. It follows from Lemma 5 that $\tilde{\alpha}_n(q) = 1$ for all q. Meanwhile, it follows again from Proposition 2 that $\alpha_N(q)$ is constant in q, and because $\alpha_N(p) < 1$, $\lim_{q \to \tilde{p}_n^* +} \alpha_N(q) < 1$. In the case where $n < N$, because $p_n^* < \tilde{p}_n^*$, it follows from Proposition 1 that $\alpha_n(\tilde{p}_n^*) < 1$.

Thus, we have that both when $k_n > \tilde{\beta}$ and when $k_n \leq \tilde{\beta}$, there exists some $\hat{p} < p$ such that $\tilde{\alpha}_n(\hat{p}) > \alpha_n(\hat{p})$ and $\tilde{\alpha}_n$, α_n satisfy (ODE) on $[\hat{p}, p]$, for their respective value of β. Thus, there exists a $q \in [\hat{p}, p]$ such that $\alpha_n(q) = \tilde{\alpha}_n(q)$ and $\alpha_n'(q) \geq \tilde{\alpha}_n'(q)$. It follows from (ODE) that in order for the above two conditions to hold, it must be that

$$X \equiv (\beta - \tilde{\beta})(\frac{1-\alpha_n(q)}{\alpha_n(q)})(1-q) + \frac{V_{q^i,n+1} - \tilde{V}_{q^i,n+1}}{\alpha_n(q)} \geq 0. \tag{34}$$

where V and $\tilde{V}$ denote the value functions under β and $\tilde{\beta}$, respectively. First consider the case where $n = N$. Then $V_{q^i,n+1} = V_{\tilde{q}^i,n+1} = 0$, and thus $X < 0$, contradicting (34).

Next, consider the case where $n < N$. First suppose that $\alpha_{n+1}(q^i) = 1$. It follows from the inductive assumption that $\tilde{\alpha}_{n+1}(q^i) = 1$. Thus, by Lemma 5, $V_{q^i,n+1} = \frac{k_{n+1}q^i}{N-n} = \tilde{V}_{q^i,n+1}$. Again this implies that $X < 0$, contradicting (34). Now, suppose that $\alpha_{n+1}(q^i) < 1$. It then follows from Lemma 2 that

$$V_{q^i,n+1} = V_{q^i,n+1}(\delta_0) = k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i).$$

Furthermore,

$$\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_0) = k_{n+1}\tilde{\alpha}_{n+1}(q^i) - \tilde{\beta}(1-q^i).$$

Thus, recalling that $q^i = \alpha_{n+1}(q) + (1-\alpha_{n+1}(q))q$, we have

$$V_{q^i,n+1} - \tilde{V}_{q^i,n+1} \leq k_{n+1}(\alpha_{n+1}(q^i) - \tilde{\alpha}_{n+1}(q^i)).$$

Substituting this into the above expression for X, we obtain

$$X \leq \frac{k_{n+1}(\alpha_{n+1}(q^i) - \tilde{\alpha}_{n+1}(q^i))}{\alpha_n(q)} < 0.$$

where the strict inequality follows from the inductive assumption that $\alpha_{n+1}(q^i) < \tilde{\alpha}_{n+1}(q^i))$. Again, this is a contradiction of (34).

Next, let us establish part (b). Let $\tilde{\lambda} > \lambda > 0$, and let α, $\tilde{\alpha}$ denote the equilibria under λ and $\tilde{\lambda}$, respectively, fixing all other parameters. We begin by showing that $\tilde{\alpha}_n(p) = \tilde{\alpha}_n(p)$ for any p and n. Fix an n and assume inductively that if $n < N$, $\alpha_{n+1}(p) = \tilde{\alpha}_{n+1}(p)$ for all p on-path. Let V, $\tilde{V}$ denote the value functions under the equilibria associated with λ and $\tilde{\lambda}$, respectively. Note that $V_{p,n+1} = \tilde{V}_{p,n+1}$ for all p on-path. In the case where $n = N$, $V_{p,n+1} = \tilde{V}_{p,n+1} = 0$, and thus this holds trivially. In the case where $n < N$, this follows from the inductive assumption.

By Lemma 5, α_n and $\tilde{\alpha}_n$ must both be a solution to (P) at all (p, n) on-path, which does not depend on λ. By Theorem 1, the solution to (P) is unique, and thus $\alpha_n(p) = \tilde{\alpha}_n(p)$ at all (p, n) on-path. Now fixing any p and n, let $p(t)$ and $\tilde{p}(t)$ denote the common beliefs under λ and $\tilde{\lambda}$, respectively. It follows from (2) that $p(t) > \tilde{p}(t)$ for all $t > 0$. Thus, because $\alpha_n(p)$ and $\tilde{\alpha}_n(p)$ are both weakly decreasing in p (Proposition 3), it follows that $\alpha_n(p(t)) \leq \tilde{\alpha}_n(p(t))$. Furthermore, since $\tilde{\alpha}(p)$ is strictly decreasing in p (Proposition 3) whenever $\alpha_n(p) < 1$ and $k_N > \beta$, it follows that $\alpha_n(p(t)) < \alpha_n(\tilde{p(t)})$ in this case.

Finally, let us establish part (c). Let α and $\tilde{\alpha}$ denote the equilibria under N and $N+1$ firms, respectively, fixing all other parameters. We begin by showing that for all p, $\alpha_n(p) \geq \tilde{\alpha}_n(p)$, and $\alpha_n(p) > \tilde{\alpha}_n(p)$ when $\alpha_n(p) < 1$. To this end, fix an $n \in \{1, ..., N\}$ and assume inductively that the claim holds for $n+1$ whenever $n < N$.

We begin by showing that $\tilde{\alpha}_n(p) = 1$ implies that $\alpha_n(p) = 1$. Suppose that $\tilde{\alpha}_n(p) = 1$. By Proposition 1, $\beta > k_n$ and $p < \tilde{p}_n^* \equiv \frac{\beta - k_n}{\beta - k_n/(N+1-n)}$. Because $p_n^* \equiv \frac{\beta - k_n}{\beta - k_n/(N-n)} > \tilde{p}_n^*$, it follows from Proposition 1 that $\alpha_n(p) = 1$.

Now consider the case where $\tilde{\alpha}_n(p) < 1$. We wish tot show that $\tilde{\alpha}_n(p) < \alpha_n(p)$. To this end, we begin by making the following observation:

$$\text{If } \alpha_n \text{ and } \tilde{\alpha}_n \text{ both satisfy (ODE) at } q, \text{ and } \alpha_n(q) = \tilde{\alpha}_n(q), \text{ then } \alpha_n'(q) > \tilde{\alpha}_n'(q). \tag{35}$$

Let us now establish this. Note first that for α_n and $\tilde{\alpha}_n$ to both satisfy (ODE) at q, given that $\alpha_n(q) = \tilde{\alpha}_n(q)$, the following must hold:

$$\alpha_n'(q) = \frac{-1}{k_n(1-q)\alpha_n(q)} \frac{N-n}{N-n+1}(k_n\alpha_n(q) - V_{q^i,n+1} - \beta(1-\alpha_n(q))(1-q))$$

$$\tilde{\alpha}_n'(q) = \frac{-1}{k_n(1-q)\alpha_n(q)} \frac{N-n+1}{N-n+2}(k_n\alpha_n(q) - \tilde{V}_{q^i,n+1} - \beta(1-\alpha_n(q))(1-q)),$$

where V and $\tilde{V}$ denote the value functions under the equilibria with N and $N+1$ total firms, respectively. Note that if $n = N$, $\alpha_n'(q) = 0$. Meanwhile, by Proposition 3, $\tilde{\alpha}_n'(q) < 0$. Thus, $\tilde{\alpha}_n'(q) < \alpha_n(q)$ must hold. Next, consider the case where $n < N$. We begin by obsevrting that $V_{q^i,n+1} > \tilde{V}_{q^i,n+1}$. To see why this must hold, first consider the case where $\tilde{\alpha}_{n+1}(q^i) = 1$. It then follows from the inductive assumption that $\alpha_n(q^i) = 1$. Then, by Lemma 5,

$$\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_\infty) = \frac{k_{n+1}q^i}{N-n} < \frac{k_{n+1}q^i}{N-n-1} = V_{q^i,n+1}(\delta_\infty) = V_{q^i,n+1}.$$

Next, consider the case where $\tilde{\alpha}_n(q^i) < 1$. In this case, it follows from Lemma 2 that

$$\begin{aligned}\tilde{V}_{q^i,n+1} = \tilde{V}_{q^i,n+1}(\delta_0) = k_{n+1}\tilde{\alpha}_{n+1}(q^i) - \beta(1-q^i) < k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i) \\ = V_{q^i,n+1}(\delta_0) \leq V_{q^i,n+1},\end{aligned}$$

where the strict inequality follows from the inductive assumption made above. Examining the two ODEs listed above, since by Proposition 3, $\alpha_n'(q) \leq 0$, it follows that $\tilde{\alpha}_n'(q) < \alpha_n'(q)$.

Now, assume by contradiction that $\alpha_n(p) \leq \tilde{\alpha}_n(p)$. We begin by showing that there exists a $q^* < p$ such that $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$. First consider the case where $k_n \geq \beta$. Then, by Proposition 2,

$$\lim_{q \to 0+} \alpha_n(q) = \lim_{q \to 0+} \tilde{\alpha}_n(q) = \frac{\beta}{k_n}.$$

Then, by the continuous differentiability of α_n and $\tilde{\alpha}_n$ on $(0,p)$, it follows from Equation 35 that for some $q^* < p$ sufficiently small $\alpha_n(q^*) > \tilde{\alpha}_n(q^*)$. Next, consider the case where $k_n < \beta$, and let $p_n^* \equiv \frac{\beta - k_n}{\beta/(N-n+1)-k_n}$. Note by Proposition 1 that $\alpha_n(p_n^*) = 1$. Meanwhile, because $p_n^* < \tilde{p}_n^* \equiv \frac{\beta - k_n}{\beta/(N-n+2)-k_n}$, it follows from Proposition 1 that $\tilde{\alpha}_n(p_n^*) < 1$, and thus, we have for $q^* = p_n^*$, $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$.

Since $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$ and $\tilde{\alpha}_n(p) \geq \alpha_n(p)$, by the continuous differentiability of α on $[q^*, p]$, there must exist some $q \in (q^*, p]$ such that $\alpha_n(q) = \tilde{\alpha}_n(q)$ and $\alpha_n'(q) \leq \tilde{\alpha}_n'(q)$. However, this is a contradiction of (35).

Now fixing any p and n, let $p(t)$ and $\tilde{p}(t)$ denote the common beliefs under N and $N+1$ firms, respectively. We wish to show that on some interval $[0, \bar{t}]$, where $\bar{t} > 0$, $\alpha_n(p(t)) \geq \tilde{\alpha}_n(\tilde{p}(t))$ is weakly increasing in t, and strictly so whenever $\alpha_n(p(t)) < 1$. First consider the case where $\alpha_n(p(t)) = 1$. In this case, the statement holds trivially. Next, consider the case where $\alpha_n(p) < 1$. It follows from the above that $\alpha_n(p) > \tilde{\alpha}_n(p)$. Now note that it follows from (2) that $\lim_{t \to 0+} p(t) - \tilde{p}(t) = 0$. Since $\alpha_n(p(t))$ and $\tilde{\alpha}_n(\tilde{p}(t))$ are both continuous in t (Lemma 4), it follows that for some $\bar{t} > 0$, $\alpha_n(p(t)) > \tilde{\alpha}_n(\tilde{p}(t))$ for all $t \in [0, \bar{t}]$. $\square$

Appendix F Proofs: heterogenous learning abilities

Here, we consider the extended model presented in Section 7. The objective is to establish Proposition 6. This proof will require extending certain results established in the baseline model to the extended model.

Regarding Lemmas 1-4, I will take for granted that these hold under the extended model. Formal proofs of this are omitted as all proofs presented under the baseline model will apply to the extended setting as well.

Next, I establish that Proposition 1 holds under the extended model. This claim is presented as Proposition 1′. In the analysis below, I let $V^i_{p,n}$ denote firm i's value.

Proposition 1′. *For all s, there exists a $p^{i*} \in (0,1]$ such that at any p on-path, $\alpha^i_1(p) = 1$ if and only if the following two conditions hold:*

1. $k_1 \leq \beta$
2. $p \leq p^{i*}$

Furthermore, $p^{j*} > p^{i*}$ whenever $\lambda^j > \lambda^i$ and $n > 1$.

Proof. Fix an i. Suppose that $k_1 \leq \beta$. By identical reasoning as Proposition 1, for all $q < \frac{\beta - k_1}{k_1}$, $\alpha^i_1(q) = 1$. Let

$$p^{i*} \equiv sup\{p|\alpha^i_1(p) = 1 \text{ for all } q < p\}.$$

It follows by definition that $\alpha^i_1(p) = 1$ for all $p \leq p^{i*}_1$.

Next, we will show that $\alpha^i_1(q) < 1$ whenever $k_1 > \beta$ or $p > p^{i*}_1$. Suppose not by contradiction. First, consider the case where $k_1 > \beta$ and $\alpha^i_1(p) = 1$ for some p. Then we have that

$$V^i_{p,1}(\delta_0) = k_1 p + (k_1 - \beta)(1-p) > k_1 p \leq V^i_{p,1}(\delta_\infty)$$

Thus, i can profitably deviate at p. Contradiction. Next, consider the case where $q > p^{i*}_n$ and $\alpha^i_1(p) = 1$. In this case, a contradiction follows from identical reasoning to what is presented in Proposition 1.

Finally, we show that $p^{j*} > p^{i*}$ whenever $\lambda^j > \lambda^i$. Suppose by contradiction that $p^{j*} \leq p^{i*}$. Note that because j is truth telling at $(p^{j*}_S, n = 1)$, $V^j_{p^{j*}_1,1}(\delta_\infty) \geq V^j_{p^{j*},1}(\delta_0)$. Furthermore, because $p^{j*} \leq p^{i*}$, i is also truthful at $(p^{j^*}_n, n = 1)$. Thus,

$$V^j_{p^{j*}_1,1}(\delta_0) = V^i_{p^{j*}_1,1}(\delta_\infty) = k_1 - \beta(1-p).$$

Now, note that because $\lambda^j > \lambda^i$,

$$V^j_{p^{j*}_1,1}(\delta_\infty) > V^i_{p^{j*}_1,1}(\delta_\infty).$$

Combining these inequalities we have $V^i_{p^{j*}_1,1}(\delta_\infty) < V^i_{p^{j*}_1,1}(\delta_0)$. However, because $\alpha^i_1(p^{j*}) = 1$, $V^j_{p^{j*}_n,1} = V^j_{p^{j*}_n,1}(\delta_\infty)$. Contradiction. □

Next, we extend Proposition 2 to this setting. Note this entails deriving an ODE that applies to this extended model, (ODE′).

Proposition 2′. *In equilibrium, for any p on-path, if $k_1 \geq \beta$ or $p > p^{i*}$, then the following must be satisfied:*

$$\alpha_1^{i\prime}(p) = -\beta - \frac{\sum_{j\neq i} \frac{\lambda^j}{\alpha_1^j(p)}}{\sum_j \lambda^j(1-p)}[\alpha^i(p) - \beta(1-p)]. \tag{ODE′}$$

In addition, $\lim_{p\to 0+} \alpha_1^i(p) = \beta/k_1$ *must hold if* $k_1 > \beta$, *and* $\lim_{p\to p^{i*}+} \alpha_1^i(p) = 1$ *if* $k_1 \leq \beta$.

Proof. Let us first establish that (ODE′) must hold under the conditions specified.

When $k_1 \geq \beta$ or $p > p^{i*}$, it follows from Proposition 1′ that $\alpha_1^i(p(t)) < 1$. It then follows from Lemma 2 that there exsits an $\varepsilon > 0$ such that for all $\Delta \in (0, \varepsilon)$,

$$\frac{V_{p,1}^i(\delta_\Delta) - V_{p,1}^i(\delta_0)}{\Delta} = 0.$$

Recall that $V_{p,1}^i(\delta_0) = k_1\alpha_1^i(p) - \beta(1-p)$. Meanwhile,

$$V_{p,1}^i(\delta_\Delta) = \int_0^\Delta k_1\alpha_1^i(p(s))\Psi^i(s)ds + (1 - \sum_j \lim_{s\to\Delta-} \Psi^j(s))[k_1\alpha_1(p(\Delta)) - \beta(1-p(\Delta))],$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p,1}$. Specifically, for all $s > 0$,

$$\Psi^i(s) = p\lambda^i \int_0^s e^{-\prod_{j\in S}\lambda^j r} \prod_{j\neq i}(1 - F_{p,1}^i(r)))dr$$

and for $j \neq i$,

$$\Psi^j(s) = p\int_0^s e^{-\prod_{k\neq j}\lambda^k r} \prod_{k\neq i\neq j}(1 - F_{p,1}^k(r))d(-e^{-\lambda^j r}(1 - F_{p,1}^j(r))) \\ +(1-p)\int_0^s \prod_{k\neq i\neq j}(1 - F_{p,1}^k(r))dF_{p,1}^j(r).$$

Substituting these two expressions into the above equation for $V_{p,1}^i(\delta_0)$ and following the same sequence of steps in Proposition 2 yields (ODE′).

The two limit conditions are established by the same reasoning presented in Proposition 2. □

Proof of Proposition 6. Fix any (i, j) such that $\lambda^i > \lambda^j$. We want to show that $\alpha_1^i(p(t)) \leq \alpha_i^j(p(t))$ and that $\alpha_1^i(p(t)) < \alpha_i^j(p(t))$ whenever $\alpha_1^i(p(t)) < 1$. First suppose $\alpha_1^i(p) = 1$. In this case, $\alpha_1^i(p) \geq \alpha_1^j(p)$ is trivially satisfied.

Next, suppose $\alpha_1^i(p) < 1$. We want to show that $\alpha_1^i(p) > \alpha_1^j(p)$. Suppose by contradiction that $\alpha_1^i(p) \leq \alpha_1^j(p)$. First consider the case where $k_1 < \beta$. Then, let

$$q^* \equiv \inf\{q | \alpha_1^j(p) < 1 \text{ and } \alpha_1^j(p) < \alpha_1^i(p)\}.$$

Because the α_1^i are continuous, it follows from Proposition 1′, and the assumption that $\alpha_1^i(p) \leq \alpha_1^j(p)$, that $q^* < p$ exists. Again, by continuity, $\alpha_1^j(q^*) = \alpha_1^i(q^*)$. It then follows from (ODE′) that $\alpha_1^{j\prime}(q^*) < \alpha_1^{i\prime}(q^*)$. But this implies that for some $q > q^*$, $\alpha_1^j(q^*) > \alpha_1^i(q^*)$. Contradiction.

Next, consider the case where $k_1 \geq \beta$. Recall by Proposition 2′ that $\lim_{p\to 0+} \alpha_1^i(p) = \lim_{p\to 0+} \alpha_1^j(p)$. Thus, there exists some $q \in (0, p]$ such that $\alpha_1^i(p) \leq \alpha_1^j(p)$ and $\alpha_1^{i\prime}(p) \leq \alpha_1^{j\prime}(p)$. However, it again follows from (ODE′) that $\alpha_1^{i\prime}(p) > \alpha_1^{j\prime}(p)$. Contradiction.

$\square$